ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$6.(3r)^6 \le \frac{r_a^6+r_b^6}{r_c}+\frac{r_b^6+r_c^6}{r_a}+\frac{r_c^6+r_a^6}{r_b} \le 2.\left(\frac{9}{r}\right)^3\left(2187\left(\frac{R}{2}\right)^8-2186r^8\right)$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} r_a^2 = (4R+r)^2-2s^2 \overset{\substack{\text{Euler} \\ \text{and} \\ \text{Gerretsen + Euler}}}{\le} \frac{81R^2}{4}-27Rr \Rightarrow \sum_{cyc} r_a^2 \le \frac{27R(3R-4r)}{4} \to (1)$$

$$\sum_{cyc} r_a^4 = \left(\sum_{cyc} r_a^2\right)^2 - 2rs^2(s^2-8Rr-2r^2) \overset{\substack{\text{via (1),Gerretsen} \\ \text{and} \\ \text{Gerretsen + Euler}}}{\le} \frac{729R^2(3R-4r)^2}{16}$$

$$- r.27Rr(8Rr-7r^2) \Rightarrow \sum_{cyc} r_a^4 \le \frac{27R(243R^3-648R^2r+304Rr^2+112r^3)}{16}$$

$$\to (2)$$

$$\text{Now, } \sum_{cyc} \frac{r_b^6+r_c^6}{r_a} = \frac{\left(\sum_{cyc} r_a^2 - r_a^2\right)\left(r_b^4+r_c^4-r_b^2r_c^2\right)}{r_a}$$

$$= \left(\sum_{cyc} r_a^2\right)\sum_{cyc}\frac{\sum_{cyc} r_a^4 - r_a^4 - r_b^2r_c^2}{r_a} - \sum_{cyc} r_a\left(r_b^4+r_c^4-r_b^2r_c^2\right)$$

$$\overset{\substack{\text{via (2)} \\ \text{and} \\ \text{A-G}}}{\le} \left(\sum_{cyc} r_a^2\right)\left(\frac{27R(243R^3-648R^2r+304Rr^2+112r^3)}{16r} - 2\sum_{yc}\frac{r_a^2r_br_c}{r_a}\right) - \sum_{yc} r_ar_b^2r_c^2$$

$$= \left(\sum_{cyc} r_a^2\right)\left(\frac{27R(243R^3-648R^2r+304Rr^2+112r^3)}{16r} - 6rs^2\right)$$

$$- rs^4 \overset{\substack{\text{Gerretsen + Euler} \\ \text{and} \\ \text{Mitrinovic}}}{\le} \left(\sum_{cyc} r_a^2\right)\left(\frac{27R(243R^3-648R^2r+304Rr^2+112r^3)}{16r}\right.$$

$$\left. - 6r.\frac{27Rr}{2}\right) - r.729r^4$$

$$= \left(\sum_{cyc} r_a^2\right).\frac{27R(243R^3-648R^2r+304Rr^2+48r^3)}{16r} \overset{\text{via (1)}}{\le} \frac{27R(3R-4r)}{4}.\frac{27R(243R^3-648R^2r+304Rr^2+48r^3)}{16r}$$

$$- 729r^5$$

$$\left(\because 243R^3-648R^2r+304Rr^2+48r^3 = (R-2r)(243R^2-162Rr-20r^2)+24r^3 \overset{\text{Euler}}{\ge} 24r^3 > 0\right)$$

$$= \frac{729\left(R^2(3R-4r)(243R^3-648R^2r+304Rr^2+48r^3)-64r^6\right)}{64r} \overset{?}{\leq} 2.\left(\frac{9}{r}\right)^3\left(2187\left(\frac{R}{2}\right)^8 - 2186r^8\right) = \frac{729\left(2187R^8-2186(256r^8)\right)}{128r^3}$$

$$\Leftrightarrow 2187t^8 - 1458t^6 + 5832t^5 - 7008t^4 + 2048t^3 + 512t^2 - 559488 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(2187t^7 + 4374t^6 + 7290t^5 + 20412t^4 + 33816t^3 + 69680t^2 + 139872t + 279744) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\therefore \frac{r_a^6+r_b^6}{r_c} + \frac{r_b^6+r_c^6}{r_a} + \frac{r_c^6+r_a^6}{r_b} \leq 2.\left(\frac{9}{r}\right)^3\left(2187\left(\frac{R}{2}\right)^8 - 2186r^8\right)$$

$$\text{Again}, \sum_{cyc}\frac{r_b^6+r_c^6}{r_a} \overset{\text{Chebyshev}}{\geq} \frac{1}{3}\left(\sum_{cyc}(r_b^6+r_c^6)\right)\left(\sum_{cyc}\frac{1}{r_a}\right) \left(\because \text{WLOG assuming } a \geq b \geq c \Rightarrow r_b^6 + r_c^6 \leq r_c^6 + r_a^6 \leq r_a^6 + r_b^6 \text{ and } \frac{1}{r_a} \leq \frac{1}{r_b} \leq \frac{1}{r_c}\right) = \frac{2}{3}\sum_{cyc} r_a^6 \overset{A-G}{\geq} 2rs^4$$

$$\overset{\text{Mitrinovic}}{\geq} 2r(729r^4) \therefore \frac{r_a^6+r_b^6}{r_c} + \frac{r_b^6+r_c^6}{r_a} + \frac{r_c^6+r_a^6}{r_b} \geq 6.(3r)^6 \text{ and so}, 6.(3r)^6 \leq \frac{r_a^6+r_b^6}{r_c} + \frac{r_b^6+r_c^6}{r_a} + \frac{r_c^6+r_a^6}{r_b} \leq 2.\left(\frac{9}{r}\right)^3\left(2187\left(\frac{R}{2}\right)^8 - 2186r^8\right) \text{ (QED)}$$