ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$18r \le \frac{r_a^4+r_b^4}{r_c^3}+\frac{r_b^4+r_c^4}{r_a^3}+\frac{r_c^4+r_a^4}{r_b^3} \le \frac{18}{r^9}\left(19683\left(\frac{R}{2}\right)^{10}-19682r^{10}\right)$$

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$$\sum_{cyc}\frac{1}{r_a^3}=\sum_{cyc}\frac{(s-a)^3}{r^3s^3}=\frac{1}{r^3s^3}\left(\left(\sum_{cyc}(s-a)\right)^3-3\prod_{cyc}(s-b+s-c)\right)$$

$$=\frac{s^3-12Rrs}{r^3s^3}\overset{\text{Mitrinovic}}{\le}\frac{\frac{27R^2}{4}-12Rr}{r^3.27r^2}\Rightarrow\sum_{cyc}\frac{1}{r_a^3}\le\frac{9R^2-16Rr}{36r^5}\to(1)$$

$$\sum_{cyc}r_a^4=((4R+r)^2-2s^2)^2-2s^2(s^2-8Rr-2r^2)$$

$$=(4R+r)^4+2s^4-4s^2(4R+r)^2+4rs^2(4R+r)\overset{\substack{\text{Trucht}\\ \text{and}\\ \because s^2\ge 12Rr+3r^2}}{\le}$$

$$(4R+r)^4+\frac{2(4R+r)^4}{9}-12r(4R+r)^3+\frac{4r(4R+r)^3}{3}$$

$$=\frac{(44R-85r)(4R+r)^3}{9}\overset{\text{Euler}}{\le}\frac{(44R-85r)\left(\frac{9R}{2}\right)^3}{9}$$

$$\therefore\sum_{cyc}\frac{r_b^4+r_c^4}{r_a^3}=\left(\sum_{cyc}r_a^4\right)\left(\sum_{cyc}\frac{1}{r_a^3}\right)-\sum_{cyc}r_a\le$$

$$\frac{(44R-85r)\left(\frac{9R}{2}\right)^3}{9}.\left(\sum_{cyc}\frac{1}{r_a^3}\right)-(4R+r)\overset{\substack{\text{via }(1)\\ \text{and}\\ \text{via Euler}}}{\le}$$

$$\frac{(44R-85r)(81R^3)}{8}.\frac{9R^2-16Rr}{36r^5}-9r$$

$$\therefore\frac{r_a^4+r_b^4}{r_c^3}+\frac{r_b^4+r_c^4}{r_a^3}+\frac{r_c^4+r_a^4}{r_b^3}\le 9.\frac{R^3(44R-85r)(9R^2-16Rr)-32r^6}{32r^5}\to(2)$$

$$\text{Again, }\frac{18}{r^9}\left(19683\left(\frac{R}{2}\right)^{10}-19682r^{10}\right)\overset{\text{Euler}}{\ge}\frac{18}{r^9}\left(\frac{19683R^6.16r^4-1024.19682r^{10}}{1024}\right)$$

$$\Rightarrow\frac{18}{r^9}\left(19683\left(\frac{R}{2}\right)^{10}-19682r^{10}\right)\ge 9.\frac{19683R^6-64.19682r^6}{32r^5}\to(3)\therefore(2),(3)\Rightarrow$$

$$\text{in order to prove : } \sum_{cyc} \frac{r_b^4 + r_c^4}{r_a^3} \le \frac{18}{r^9}\left(19683\left(\frac{R}{2}\right)^{10} - 19682r^{10}\right), \text{ it suffices}$$

$$\text{to prove : } \frac{19683R^6 - 64.19682r^6}{32r^5} \ge \frac{R^3(44R - 85r)(9R^2 - 16Rr) - 32r^6}{32r^5}$$

$$\Leftrightarrow 19287t^6 + 1469t^5 - 1360t^4 - 1259616 \overset{?}{\ge} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left(19287t^5 + 40043t^4 + 78726t^3 + 157452t^2 + 314904t + 629808\right) \overset{?}{\ge} 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\ge} 2 \therefore \frac{r_a^4 + r_b^4}{r_c^3} + \frac{r_b^4 + r_c^4}{r_a^3} + \frac{r_c^4 + r_a^4}{r_b^3} \le \frac{18}{r^9}\left(19683\left(\frac{R}{2}\right)^{10} - 19682r^{10}\right)$$

$$\text{Also, } \frac{r_a^4 + r_b^4}{r_c^3} + \frac{r_b^4 + r_c^4}{r_a^3} + \frac{r_c^4 + r_a^4}{r_b^3} \overset{\text{Radon}}{\ge} \frac{\left(2\sum_{cyc} r_a\right)^4}{\left(2\sum_{cyc} r_a\right)^3} = 2(4R + r) \overset{\text{Euler}}{\ge} 18r$$

$$\text{and so, } 18r \le \frac{r_a^4 + r_b^4}{r_c^3} + \frac{r_b^4 + r_c^4}{r_a^3} + \frac{r_c^4 + r_a^4}{r_b^3} \le \frac{18}{r^9}\left(19683\left(\frac{R}{2}\right)^{10} - 19682r^{10}\right)$$

$$\forall \Delta ABC, " = " \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$