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In ΔABC the following relationship holds:

$$a\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+b\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+c\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\geq 12r$$

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Solution by Daniel Sitaru-Romania

$$a\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+b\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+c\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)=$$

$$=\sum_{cyc}a\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=\sum_{cyc}a\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=$$

$$=\sum_{cyc}a\left(\frac{\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{B}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}\right)=\sum_{cyc}a\left(\frac{\sin\left(\frac{B+C}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}}\right)=$$

$$=\sum_{cyc}a\left(\frac{\sin\left(\frac{\pi-A}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}}\right)=\sum_{cyc}a\left(\frac{\cos\left(\frac{A}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}}\right)\overset{AM-GM}{\geq}3\sqrt[3]{\frac{abc}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}}=$$

$$=3\sqrt[3]{\frac{4Rrs}{\frac{s}{4R}}}=3\sqrt[3]{16R^2r}\overset{EULER}{\geq}3\sqrt[3]{16\cdot 4r^2\cdot r}=12r$$

Equality holds for $a=b=c$.