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In ΔABC the following relationship holds:

$$(b+c)\tan\frac{A}{2}+(c+a)\tan\frac{B}{2}+(a+b)\tan\frac{C}{2}\geq 12r$$

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$$(b+c)\tan\frac{A}{2}+(c+a)\tan\frac{B}{2}+(a+b)\tan\frac{C}{2}=$$

$$=\sum_{cyc}(b+c)\tan\frac{A}{2}=2R\sum_{cyc}(\sin B+\sin C)\tan\frac{A}{2}=$$

$$=4R\sum_{cyc}\sin\frac{B+C}{2}\cos\frac{B-C}{2}\tan\frac{A}{2}=4R\sum_{cyc}\sin\frac{\pi-A}{2}\cos\frac{B-C}{2}\tan\frac{A}{2}=$$

$$=4R\sum_{cyc}\cos\frac{A}{2}\cos\frac{B-C}{2}\cdot\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=4R\sum_{cyc}\sin\frac{A}{2}\cos\frac{B-C}{2}\overset{AM-GM}{\geq}$$

$$\geq 12R\cdot\sqrt[3]{\prod_{cyc}\sin\frac{A}{2}\cdot\prod_{cyc}\cos\frac{B-C}{2}}=12R\sqrt[3]{\frac{r}{4R}\cdot\frac{s^2+r^2+2Rr}{8R^2}}\geq$$

$$\overset{GERRETSEN}{\geq}12R\sqrt[3]{\frac{r}{4R}\cdot\frac{16Rr-5r^2+r^2+2Rr}{8R^2}}=12R\sqrt[3]{\frac{r(18Rr-4r^2)}{32R^3}}\geq$$

$$\overset{EULER}{\geq}12\sqrt[3]{\frac{r(36r^2-4r^2)}{32}}=12r$$

Equality holds for $a=b=c$.