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In ΔABC the following relationship holds:

$$a^2\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+b^2\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+c^2\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\geq 24\sqrt{3}r^2$$

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Solution by Daniel Sitaru-Romania

$$a^2\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+b^2\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+c^2\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)=$$

$$=\sum_{cyc}a^2\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)\overset{AM-GM}{\geq}2\sum_{cyc}a^2\sqrt{\tan\frac{B}{2}\cdot\tan\frac{C}{2}}\overset{AM-GM}{\geq}$$

$$\geq 6\sqrt[3]{\prod_{cyc}a^2\sqrt{\tan\frac{B}{2}\cdot\tan\frac{C}{2}}}=6\sqrt[3]{(abc)^2\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}=$$

$$=6\sqrt[3]{16R^2r^2s^2\cdot\frac{r}{s}}=12r\sqrt[3]{2R^2s}\overset{EULER}{\geq}12r\sqrt[3]{8r^2s}\overset{MITRINOVIC}{\geq}$$

$$=12r\sqrt[3]{8r^2\cdot 3\sqrt{3}r}=12r^2\sqrt[3]{\left(2\sqrt{3}\right)^3}=24\sqrt{3}r^2$$

Equality holds for $a=b=c$.