ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$(b^2+c^2)\tan\frac{A}{2}+(c^2+a^2)\tan\frac{B}{2}+(a^2+b^2)\tan\frac{C}{2}\ge 24\sqrt{3}r^2$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$(b^2+c^2)\tan\frac{A}{2}+(c^2+a^2)\tan\frac{B}{2}+(a^2+b^2)\tan\frac{C}{2}=\sum_{cyc}(b^2+c^2)\tan\frac{A}{2}\ge$$

$$\overset{AM-GM}{\geq} 2\sum_{cyc} bc\cdot\tan\frac{A}{2}\overset{AM-GM}{\geq} 2\cdot 3\sqrt[3]{(abc)^2\prod_{cyc}\tan\frac{A}{2}}=$$

$$=6\cdot\sqrt[3]{(4Rrs)^2\cdot\frac{r}{s}}=6\cdot\sqrt[3]{16R^2r^3s}=6r\cdot\sqrt[3]{16R^2s}\overset{EULER}{\geq}$$

$$\ge 6r\cdot\sqrt[3]{64r^2s}=24r\cdot\sqrt[3]{r^2s}\overset{MITRINOVIC}{\geq} 24r\cdot\sqrt[3]{3\sqrt{3}\cdot r^3}=$$

$$=24r^2\cdot\sqrt[3]{\left(\sqrt{3}\right)^3}=24\sqrt{3}r^2$$

Equality holds for $a=b=c$.