ROMANIAN MATHEMATICAL MAGAZINE

In all acute triangles $ABC, n \in \mathbb{N}$ the following relationship holds:

$$a^n(\cot B+\cot C)+b^n(\cot C+\cot A)+c^n(\cot A+\cot B)\ge 2^{n+1}\cdot 3^{\frac{n+1}{2}}\cdot r^n$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$a^n(\cot B+\cot C)+b^n(\cot C+\cot A)+c^n(\cot A+\cot B)=$$

$$=\sum_{cyc} a^n\left(\frac{\cos B}{\sin B}+\frac{\cos C}{\sin C}\right)=\sum_{cyc} a^n\cdot\frac{\sin C\cos B+\sin B\cos C}{\sin B\sin C}=$$

$$=\sum_{cyc} a^n\cdot\frac{\sin(B+C)}{\sin B\sin C}=\sum_{cyc} a^n\cdot\frac{\sin(\pi-A)}{\sin B\sin C}=\sum_{cyc} a^n\cdot\frac{\sin A}{\sin B\sin C}\overset{AM-GM}{\ge}$$

$$\ge 3\cdot\sqrt[3]{(abc)^n\cdot\frac{\sin A}{\sin B\sin C}\cdot\frac{\sin B}{\sin C\sin A}\cdot\frac{\sin C}{\sin C\sin B}}=$$

$$=3\cdot\sqrt[3]{(4RF)^n\cdot\frac{1}{\sin A\sin B\sin C}}=3\cdot\sqrt[3]{(4Rrs)^n\cdot\frac{2R\cdot 2R\cdot 2R}{abc}}=$$

$$=6R\cdot\sqrt[3]{(4Rrs)^{n-1}}\overset{EULER}{\ge}12r\sqrt[3]{(8r^2s)^{n-1}}\ge$$

$$\overset{MITRINOVIC}{\ge}12r\sqrt[3]{\left(8r^3\cdot 3\sqrt{3}\right)^{n-1}}=12r\left(2\sqrt{3}r\right)^{n-1}=$$

$$=3r\cdot 2^2\cdot 2^{n-1}\cdot 3^{\frac{n-1}{2}}\cdot r^{n-1}=2^{n+1}\cdot 3^{\frac{n+1}{2}}\cdot r^n$$

Equality holds for $a=b=c$.