ROMANIAN MATHEMATICAL MAGAZINE

In all acute triangles ABC the following relationship holds:

$$(\cot B+\cot C)\tan\frac{A}{2}+(\cot C+\cot A)\tan\frac{B}{2}+(\cot A+\cot B)\tan\frac{C}{2}\geq 2$$

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Solution by Daniel Sitaru-Romania

$$(\cot B+\cot C)\tan\frac{A}{2}+(\cot C+\cot A)\tan\frac{B}{2}+(\cot A+\cot B)\tan\frac{C}{2}=$$

$$=\sum_{cyc}(\cot B+\cot C)\tan\frac{A}{2}=\sum_{cyc}\left(\frac{\cos B}{\sin B}+\frac{\cos C}{\sin C}\right)\tan\frac{A}{2}=$$

$$=\sum_{cyc}\frac{\cos B\sin C+\cos C\sin B}{\sin B\sin C}\cdot\tan\frac{A}{2}=\sum_{cyc}\frac{\sin(B+C)}{\sin B\sin C}\cdot\tan\frac{A}{2}=$$

$$=\sum_{cyc}\frac{\sin(\pi-A)}{\sin B\sin C}\cdot\tan\frac{A}{2}=\sum_{cyc}\frac{\sin A}{\sin B\sin C}\cdot\tan\frac{A}{2}\overset{AM-GM}{\geq}$$

$$\geq 3\sqrt[3]{\prod_{cyc}\frac{1}{\sin A}\cdot\prod_{cyc}\tan\frac{A}{2}}=3\sqrt[3]{\frac{2R^2}{sr}\cdot\frac{r}{s}}=3\sqrt[3]{2\cdot\frac{R^2}{s^2}}\geq$$

$$\overset{MITRINOVIC}{\geq}3\sqrt[3]{\frac{2R^2}{\frac{27R^2}{4}}}=3\sqrt[3]{\frac{8}{27}}=2$$

Equality holds for $a=b=c$.