ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$h_a\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+h_b\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+h_c\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\geq 6\sqrt{3}r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$h_a\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+h_b\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+h_c\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)=$$

$$=\sum_{cyc}h_a\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=\sum_{cyc}h_a\cdot\frac{\sin\left(\frac{B+C}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}}=$$

$$=2F\sum_{cyc}\frac{\cos\frac{A}{2}}{a\cos\frac{B}{2}\cos\frac{C}{2}}\overset{AM-GM}{\geq}2F\cdot 3\sqrt[3]{\frac{1}{abc}\cdot\prod_{cyc}\frac{1}{\cos\frac{A}{2}}}=$$

$$=6F\cdot\sqrt[3]{\frac{1}{4RF}\cdot\frac{4R}{s}}=6rs\cdot\sqrt[3]{\frac{1}{Fs}}=6r\cdot\sqrt[3]{\frac{s^3}{rs^2}}=$$

$$=6r\cdot\sqrt[3]{\frac{s}{r}}\overset{MITRINOVIC}{\geq}6r\cdot\sqrt[3]{\frac{3\sqrt{3}r}{r}}=6r\sqrt[3]{\left(\sqrt{3}\right)^3}=6\sqrt{3}r$$

Equality holds for $a=b=c$.