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In ΔABC the following relationship holds:

$$min\left(\sum_{cyc} w_a\left(tan\frac{B}{2}+tan\frac{C}{2}\right),\sum_{cyc} m_a\left(tan\frac{B}{2}+tan\frac{C}{2}\right)\right)\ge 6\sqrt{3}r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

Lemma: In ΔABC the following relationship holds:

$$\sum_{cyc} h_a\left(tan\frac{B}{2}+tan\frac{C}{2}\right)\ge 6\sqrt{3}r$$

Proof:

$$\sum_{cyc} h_a\left(tan\frac{B}{2}+tan\frac{C}{2}\right)=\sum_{cyc} h_a\cdot\frac{sin\left(\frac{B+C}{2}\right)}{cos\frac{B}{2}cos\frac{C}{2}}=$$

$$=2F\sum_{cyc}\frac{cos\frac{A}{2}}{acos\frac{B}{2}cos\frac{C}{2}}\overset{AM-GM}{\ge}2F\cdot 3\sqrt[3]{\frac{1}{abc}\cdot\prod_{cyc}\frac{1}{cos\frac{A}{2}}}=6F\cdot\sqrt[3]{\frac{1}{4RF}\cdot\frac{4R}{s}}=6rs\cdot\sqrt[3]{\frac{1}{Fs}}=$$

$$=6r\cdot\sqrt[3]{\frac{s^3}{rs^2}}=6r\cdot\sqrt[3]{\frac{s}{r}}\overset{MITRINOVIC}{\ge}6r\cdot\sqrt[3]{\frac{3\sqrt{3}r}{r}}=6r\sqrt[3]{\left(\sqrt{3}\right)^3}=6\sqrt{3}r$$

Equality holds for $a=b=c$.

Back to the problem:

$$w_a\ge h_a\Rightarrow\sum_{cyc} w_a\left(tan\frac{B}{2}+tan\frac{C}{2}\right)\ge\sum_{cyc} h_a\left(tan\frac{B}{2}+tan\frac{C}{2}\right)\overset{Lemma}{\Rightarrow}\ge 6\sqrt{3}r$$

$$m_a\ge h_a\Rightarrow\sum_{cyc} m_a\left(tan\frac{B}{2}+tan\frac{C}{2}\right)\ge\sum_{cyc} h_a\left(tan\frac{B}{2}+tan\frac{C}{2}\right)\overset{Lemma}{\Rightarrow}\ge 6\sqrt{3}r$$

$$min\left(\sum_{cyc} w_a\left(tan\frac{B}{2}+tan\frac{C}{2}\right),\sum_{cyc} m_a\left(tan\frac{B}{2}+tan\frac{C}{2}\right)\right)\ge 6\sqrt{3}r$$

Equality holds for $a=b=c$.