ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$(h_b+h_c)\cdot\tan\frac{A}{2}+(h_c+h_a)\cdot\tan\frac{B}{2}+(h_a+h_b)\cdot\tan\frac{C}{2}\ge 6\sqrt{3}r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Ertan Yildirim-Turkiye

Lemma: $h_a+h_b+h_c\ge 9r$

Let $a\ge b\ge c$ then $h_a\le h_b\le h_c$ and $h_a+h_b\le h_a+h_c\le h_b+h_c$

$$\tan\frac{A}{2}\ge\tan\frac{B}{2}\ge\tan\frac{C}{2}$$

$$\therefore\sum(h_b+h_c)\tan\frac{A}{2}\overset{Chebyshev}{\ge}\frac{1}{3}\cdot 2(h_a+h_b+h_c)\cdot\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)$$

$$\overset{Lemma}{\ge}\frac{1}{3}\cdot 2\cdot 9r\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)$$

$$=6r\cdot\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)$$

$f(x)=\tan x\Rightarrow f'(x)=\sec^2 x\Rightarrow f''(x)=2\sec^2 x\cdot\tan x>0$ where $x\in\left(0,\frac{\pi}{2}\right)$

Therefore $f(x)$ is convex

$$\frac{f\left(\frac{A}{2}\right)+f\left(\frac{B}{2}\right)+f\left(\frac{C}{2}\right)}{3}\ge f\left(\frac{A+B+C}{6}\right)$$

$$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\ge 3\cdot f\left(\frac{\pi}{6}\right)=3\cdot\tan\frac{\pi}{6}=3\frac{\sqrt{3}}{3}=\sqrt{3}$$

$$\therefore\sum(h_b+h_c)\tan\frac{A}{2}\ge 6r\cdot\sum\tan\frac{A}{2}\ge 6r\cdot\sqrt{3}=6\sqrt{3}r$$