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In ΔABC the following relationship holds:

$$a\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)+b\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)+c\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)\ge 36r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$a\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)+b\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)+c\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)=\sum_{cyc}a\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)=$$

$$=\sum_{cyc}a\left(\frac{\cos\frac{B}{2}}{\sin\frac{B}{2}}+\frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}\right)=\sum_{cyc}a\cdot\frac{\cos\frac{B}{2}\sin\frac{C}{2}+\cos\frac{C}{2}\sin\frac{B}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}=$$

$$=\sum_{cyc}a\cdot\frac{\sin\left(\frac{B+C}{2}\right)}{\sin\frac{B}{2}\sin\frac{C}{2}}=\sum_{cyc}a\cdot\frac{\sin\left(\frac{\pi-A}{2}\right)}{\sin\frac{B}{2}\sin\frac{C}{2}}=\sum_{cyc}a\cdot\frac{\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}\overset{AM-GM}{\ge}$$

$$\ge 3\cdot\sqrt[3]{abc\cdot\prod_{cyc}\cos\frac{A}{2}\cdot\left(\prod_{cyc}\sin\frac{A}{2}\right)^{-2}}=3\cdot\sqrt[3]{abc\cdot\frac{s}{4R}\cdot\left(\frac{r}{4R}\right)^{-2}}=$$

$$=3\cdot\sqrt[3]{abc\cdot\frac{16R^2s}{4Rr^2}}=3\cdot\sqrt[3]{4Rrs\cdot\frac{16R^2s}{4Rr^2}}=3\cdot\sqrt[3]{4Rrs\cdot\frac{4Rs}{r^2}}=$$

$$=3\cdot\sqrt[3]{\frac{16R^2s^2}{r}}\overset{EULER}{\ge}3\cdot\sqrt[3]{\frac{64r^2s^2}{r}}=12\cdot\sqrt[3]{rs^2}\overset{MITRINOVIC}{\ge}12\cdot\sqrt[3]{27r^3}=36r$$

Equality holds for $a=b=c$.