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In ΔABC the following relationship holds:

$$r_a\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+r_b\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+r_c\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\geq 6\sqrt{3}r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$r_a\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+r_b\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+r_c\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)=$$

$$=\sum_{cyc}r_a\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=\sum_{cyc}r_a\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\sum_{cyc}r_a\cdot\frac{\sin\left(\frac{B+C}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}}=$$

$$=\sum_{cyc}r_a\cdot\frac{\sin\left(\frac{\pi-A}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}}=\sum_{cyc}r_a\cdot\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}\overset{AM-GM}{\geq}$$

$$=3\cdot\sqrt[3]{\prod_{cyc}r_a\cdot\prod_{cyc}\frac{1}{\cos\frac{A}{2}}}=3\cdot\sqrt[3]{rs^2\cdot\frac{4R}{s}}\overset{EULER}{\geq}3\cdot\sqrt[3]{8r^2s}\overset{MITRINOVIC}{\geq}$$

$$\geq 6\cdot\sqrt[3]{r^2\cdot 3\sqrt{3}r}=6\cdot\sqrt[3]{\left(\sqrt{3}\cdot r\right)^3}=6\sqrt{3}r$$

Equality holds for $a=b=c$.