ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$h_a\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)+h_b\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)+h_c\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)\geq 18\sqrt{3}r$$

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Solution by Daniel Sitaru-Romania

$$h_a\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)+h_b\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)+h_c\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)=$$

$$=\sum_{cyc} h_a\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)=\sum_{cyc}\frac{2F}{a}\left(\frac{\cos\frac{B}{2}}{\sin\frac{B}{2}}+\frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}\right)=$$

$$=2F\sum_{cyc}\frac{1}{a}\cdot\frac{\sin\left(\frac{B+C}{2}\right)}{\sin\frac{B}{2}\sin\frac{C}{2}}=2F\sum_{cyc}\frac{1}{a}\cdot\frac{\sin\left(\frac{\pi-A}{2}\right)}{\sin\frac{B}{2}\sin\frac{C}{2}}=2F\sum_{cyc}\frac{1}{a}\cdot\frac{\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}\geq$$

$$\overset{AM-GM}{\geq} 2F\cdot 3\sqrt[3]{\frac{1}{abc}\cdot\prod_{cyc}\cos\frac{A}{2}\cdot\left(\prod_{cyc}\sin\frac{A}{2}\right)^{-2}}=$$

$$=6F\sqrt[3]{\frac{1}{4Rrs}\cdot\frac{s}{4R}\cdot\left(\frac{4R}{r}\right)^2}=6F\sqrt[3]{\frac{1}{16R^2r}\cdot\frac{16R^2}{r^2}}=$$

$$=6rs\cdot\sqrt[3]{\frac{1}{r^3}}=6s\overset{MITRINOVIC}{\geq}6\cdot 3\sqrt{3}r=18\sqrt{3}r$$

Equality holds for $a=b=c$.