ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$min\left(\sum_{cyc}(b+c)h_a,\sum_{cyc}a(h_b+h_c)\right)\geq 36\sqrt{3}r^2$$

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$$\sum_{cyc}(b+c)h_a=\sum_{cyc}(b+c)\cdot\frac{2F}{a}=2F\sum_{cyc}\frac{b+c}{a}\overset{AM-GM}{\geq}$$

$$\geq 2F\cdot 3\sqrt[3]{\prod_{cyc}\frac{b+c}{a}}=6rs\cdot\sqrt[3]{\frac{(b+c)(c+a)(a+b)}{abc}}\overset{CESARO}{\geq}$$

$$\geq 6rs\cdot\sqrt[3]{\frac{8abc}{abc}}=6rs\sqrt[3]{8}=12rs\overset{MITRINOVIC}{\geq}12r\cdot 3\sqrt{3}r=36\sqrt{3}r^2$$

$$\sum_{cyc}a(h_b+h_c)=\sum_{cyc}a\left(\frac{2F}{b}+\frac{2F}{c}\right)=2F\sum_{cyc}a\left(\frac{1}{b}+\frac{1}{c}\right)=$$

$$=2rs\sum_{cyc}\frac{a(b+c)}{bc}\overset{AM-GM}{\geq}2rs\cdot 3\sqrt[3]{\frac{abc(b+c)(c+a)(a+b)}{abc\cdot abc}}=$$

$$=6rs\cdot\sqrt[3]{\frac{(b+c)(c+a)(a+b)}{abc}}\overset{CESARO}{\geq}$$

$$\geq 6rs\cdot\sqrt[3]{\frac{8abc}{abc}}=6rs\sqrt[3]{8}=12rs\overset{MITRINOVIC}{\geq}12r\cdot 3\sqrt{3}r=36\sqrt{3}r^2$$

Equality holds for $a=b=c$.