ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$min\left(\sum_{cyc}(b+c)r_a, \sum_{cyc}a(r_b+r_c)\right) \geq 36\sqrt{3}r^2$$

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$$\sum_{cyc}(b+c)r_a = \sum_{cyc}(b+c)\cdot\frac{F}{s-a} = F\cdot\sum_{cyc}\frac{b+c}{s-a} =$$

$$= F\cdot\sum_{cyc}\frac{2s-a}{s-a} = F\cdot\sum_{cyc}\frac{s+s-a}{s-a} = F\left(s\sum_{cyc}\frac{1}{s-a}+3\right) =$$

$$= F\left(s\cdot\frac{4R+r}{rs}+3\right) = F\left(\frac{4R+r}{r}+3\right) \overset{EULER}{\geq} F\left(\frac{4\cdot 2r+r}{r}+3\right) =$$

$$= 12F = 12rs \overset{MITRINOVIC}{\geq} 12r\cdot 3\sqrt{3}r = 36\sqrt{3}r^2$$

$$\sum_{cyc}a(r_b+r_c) = \sum_{cyc}a\left(\frac{F}{s-b}+\frac{F}{s-c}\right) = F\sum_{cyc}a\cdot\frac{s-c+s-b}{(s-b)(s-c)} =$$

$$= F\sum_{cyc}\frac{a^2}{(s-b)(s-c)} = \frac{F}{(s-a)(s-b)(s-c)}\sum_{cyc}a^2(s-a) =$$

$$= \frac{Fs}{s(s-a)(s-b)(s-c)}\cdot 4rs(R+r) = \frac{Fs}{F^2}\cdot 4F(R+r) \overset{EULER}{\geq} 4s(2r+r) =$$

$$= 12rs \overset{MITRINOVIC}{\geq} 12r\cdot 3\sqrt{3}r = 36\sqrt{3}r^2$$

Equality holds for $a=b=c$.