ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{a}{b+c}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+\frac{b}{c+a}\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+\frac{c}{a+b}\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)\geq\sqrt{3}$$

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Solution by Daniel Sitaru-Romania

$$\frac{a}{b+c}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+\frac{b}{c+a}\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+\frac{c}{a+b}\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)=$$

$$=\sum_{cyc}\frac{a}{b+c}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=\sum_{cyc}\frac{a}{b+c}\cdot\frac{\sin\left(\frac{B+C}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}}=\sum_{cyc}\frac{a}{b+c}\cdot\frac{\sin\left(\frac{\pi-A}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}}=$$

$$=\sum_{cyc}\frac{a}{b+c}\cdot\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}\overset{AM-GM}{\geq}3\cdot\sqrt[3]{\prod_{cyc}\frac{a}{b+c}\cdot\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}}=$$

$$=3\cdot\sqrt[3]{\frac{abc}{(a+b)(b+c)(c+a)}\prod_{cyc}\frac{1}{\cos\frac{A}{2}}}=3\cdot\sqrt[3]{\frac{4Rrs}{2s(s^2+r^2+2Rr)}\cdot\frac{4R}{s}}=$$

$$=6\cdot\sqrt[3]{\frac{R^2r}{s(s^2+r^2+2Rr)}}\overset{MITRINOVIC}{\geq}6\cdot\sqrt[3]{\frac{R^2r}{3\sqrt{3}r(s^2+r^2+2Rr)}}=$$

$$=\frac{6}{\sqrt{3}}\cdot\sqrt[3]{\frac{R^2}{s^2+r^2+2Rr}}\overset{GERRETSEN}{\geq}2\sqrt{3}\cdot\sqrt[3]{\frac{R^2}{4R^2+4Rr+3r^2+r^2+2Rr}}\overset{EULER}{\geq}$$

$$\geq2\sqrt{3}\cdot\sqrt[3]{\frac{R^2}{4R^2+6R\cdot\frac{R}{2}+4\left(\frac{R}{2}\right)^2}}=2\sqrt{3}\cdot\sqrt[3]{\frac{R^2}{4R^2+6R\cdot\frac{R}{2}+4\left(\frac{R}{2}\right)^2}}=$$

$$=2\sqrt{3}\cdot\sqrt[3]{\frac{1}{8}}=\sqrt{3}$$

Equality holds for: $a=b=c$.