ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{h_a}{h_b+h_c}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+\frac{h_b}{h_c+h_a}\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+\frac{h_c}{h_a+h_b}\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\geq\sqrt{3}$$

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$\forall$ A, B, C > 0, (A + B), (B + C), (C + A) form sides of a triangle ($\because$ (A + B) + (B + C) > (C + A) and analogs) $\Rightarrow \sqrt{A+B}, \sqrt{B+C}, \sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2 =$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\rightarrow(1)$$

$$\text{Now, }\forall x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}\overset{?}{\geq}$$

$$\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\rightarrow\text{true}\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\rightarrow(2)$$

$$\text{We have : }\frac{h_a}{h_b+h_c}.\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+\frac{h_b}{h_c+h_a}.\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)$$

$$+\frac{h_c}{h_a+h_b}.\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=h_a,y=h_b,z=h_c,A=\tan\frac{A}{2},B=\tan\frac{B}{2},C=\tan\frac{C}{2}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}=\sqrt{3}.\sqrt{\sum_{cyc}\left(\tan\frac{A}{2}.\tan\frac{B}{2}\right)}$$

$$=\sqrt{3}.\sqrt{\frac{1}{s^2}\sum_{cyc}r_ar_b}=\sqrt{3}.\sqrt{\frac{1}{s^2}.s^2}=\sqrt{3}$$

$$\therefore\frac{h_a}{h_b+h_c}.\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+\frac{h_b}{h_c+h_a}.\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+\frac{h_c}{h_a+h_b}.\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)$$

$\geq\sqrt{3}\ \forall\ \Delta$ ABC, " = " iff Δ ABC is equilateral (QED)