

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC , the following relationship holds :

$$\frac{a}{b+c} \cdot \left(\cot \frac{B}{2} + \cot \frac{C}{2} \right) + \frac{b}{c+a} \cdot \left(\cot \frac{C}{2} + \cot \frac{A}{2} \right) + \frac{c}{a+b} \cdot \left(\cot \frac{A}{2} + \cot \frac{B}{2} \right) \geq 3\sqrt{3}$$

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$\forall A, B, C > 0, (A+B), (B+C), (C+A)$ form sides of a triangle

$(\because (A+B) + (B+C) > (C+A)$ and analogs) $\Rightarrow \sqrt{A+B}, \sqrt{B+C}, \sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2 =$

$$\begin{aligned} 2 \sum_{\text{cyc}} (A+B)(B+C) - \sum_{\text{cyc}} (A+B)^2 &= 2 \sum_{\text{cyc}} \left(\sum_{\text{cyc}} AB + B^2 \right) - 2 \sum_{\text{cyc}} A^2 - 2 \sum_{\text{cyc}} AB \\ &= 6 \sum_{\text{cyc}} AB + 2 \sum_{\text{cyc}} A^2 - 2 \sum_{\text{cyc}} A^2 - 2 \sum_{\text{cyc}} AB \Rightarrow 4F = 2 \sqrt{\sum_{\text{cyc}} AB} \rightarrow (1) \end{aligned}$$

Now, $\forall x, y, z > 0, \sqrt{\sum_{\text{cyc}} \frac{xy}{(y+z)(z+x)}} \stackrel{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{\text{cyc}} \frac{x^2 y^2}{xy(y+z)(z+x)} \stackrel{?}{\geq} \frac{3}{4}$ (*)

Via Bergstrom, LHS of (*) $\geq \frac{(\sum_{\text{cyc}} xy)^2}{\sum_{\text{cyc}} (xy(\sum_{\text{cyc}} xy + z^2))} = \frac{(\sum_{\text{cyc}} xy)^2}{(\sum_{\text{cyc}} xy)^2 + xyz \sum_{\text{cyc}} x}$

$$\stackrel{?}{\geq} \frac{3}{4} \Leftrightarrow \left(\sum_{\text{cyc}} xy \right)^2 \stackrel{?}{\geq} 3xyz \sum_{\text{cyc}} x \rightarrow \text{true} \therefore \sqrt{\sum_{\text{cyc}} \frac{xy}{(y+z)(z+x)}} \stackrel{?}{\geq} \frac{\sqrt{3}}{2} \rightarrow (2)$$

We have : $\frac{a}{b+c} \cdot \left(\cot \frac{B}{2} + \cot \frac{C}{2} \right) + \frac{b}{c+a} \cdot \left(\cot \frac{C}{2} + \cot \frac{A}{2} \right) + \frac{c}{a+b} \cdot \left(\cot \frac{A}{2} + \cot \frac{B}{2} \right)$
 $= \frac{x}{y+z} (B+C) + \frac{y}{z+x} (C+A) + \frac{z}{x+y} (A+B)$

$$\left(x = a, y = b, z = c, A = \cot \frac{A}{2}, B = \cot \frac{B}{2}, C = \cot \frac{C}{2} \right)$$

$$= \frac{x}{y+z} \cdot \sqrt{B+C}^2 + \frac{y}{z+x} \cdot \sqrt{C+A}^2 + \frac{z}{x+y} \cdot \sqrt{A+B}^2 \stackrel{\text{Oppenheim}}{\geq}$$

$$4F \cdot \sqrt{\sum_{\text{cyc}} \frac{xy}{(y+z)(z+x)}} \stackrel{\text{via (1) and (2)}}{\geq} 2 \sqrt{\sum_{\text{cyc}} AB} \cdot \frac{\sqrt{3}}{2} = \sqrt{3} \cdot \sqrt{\sum_{\text{cyc}} \left(\cot \frac{A}{2} \cdot \cot \frac{B}{2} \right)}$$

$$= \sqrt{3} \cdot \sqrt{s^2 \sum_{\text{cyc}} \frac{1}{r_a r_b}} = \sqrt{3} \cdot \sqrt{\frac{s^2(4R+r)}{rs^2}} \stackrel{\text{Euler}}{\geq} \sqrt{3} \cdot \sqrt{8+1} = 3\sqrt{3}$$

$$\therefore \frac{a}{b+c} \cdot \left(\cot \frac{B}{2} + \cot \frac{C}{2} \right) + \frac{b}{c+a} \cdot \left(\cot \frac{C}{2} + \cot \frac{A}{2} \right) + \frac{c}{a+b} \cdot \left(\cot \frac{A}{2} + \cot \frac{B}{2} \right) \geq 3\sqrt{3}$$

$\forall \Delta ABC, "=" \text{ iff } \Delta ABC \text{ is equilateral (QED)}$