ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{m_a w_b}{w_c h_a}+\frac{w_b h_c}{h_a m_b}+\frac{h_c m_a}{m_b w_c}\le\frac{3}{8}\cdot\left(9\left(\frac{R}{r}\right)^3-64\right)$$

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$$\frac{m_a w_b}{w_c h_a}+\frac{w_b h_c}{h_a m_b}+\frac{h_c m_a}{m_b w_c}\le\frac{m_a m_b}{h_c h_a}+\frac{m_b m_c}{h_a h_b}+\frac{m_c m_a}{h_b h_c}$$

$$\le m_a m_b m_c\left(\frac{1}{h_c^2 h_a}+\frac{1}{h_a^2 h_b}+\frac{1}{h_b^2 h_c}\right)\le\frac{Rs^2}{2}\cdot\frac{c^2a+a^2b+b^2c}{8r^3s^3}$$

$$\overset{A-G}{\le}\frac{R}{16r^3s}\cdot(a^3+b^3+c^3)=\frac{2sR(s^2-6Rr-3r^2)}{16r^3s}$$

$$\overset{Gerretsen}{\le}\frac{R(4R^2-2Rr)}{8r^3}\overset{?}{\le}\frac{3}{8}\cdot\left(9\left(\frac{R}{r}\right)^3-64\right)\Leftrightarrow 23R^3+2R^2r-192r^3\overset{?}{\ge}0$$

$$\Leftrightarrow(R-2r)(23R^2+48Rr+96r^2)\overset{?}{\ge}0\to\text{true}\because R\overset{Euler}{\ge}2r$$

$$\therefore\frac{m_a w_b}{w_c h_a}+\frac{w_b h_c}{h_a m_b}+\frac{h_c m_a}{m_b w_c}\le\frac{3}{8}\cdot\left(9\left(\frac{R}{r}\right)^3-64\right)\forall\Delta ABC,$$

'' ='' iff Δ ABC is equilateral (QED)

Proof of $m_a m_b m_c\le\frac{Rs^2}{2}$

$$m_a^2m_b^2m_c^2=\frac{1}{64}(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)$$

$$\overset{(1)}{=}\frac{1}{64}\left\{-4\sum_{cyc}a^6+6\left(\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4\right)+3a^2b^2c^2\right\}$$

$$\text{Now,}\sum_{cyc}a^6=\left(\sum_{cyc}a^2\right)^3-3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$=\left(\sum_{cyc}a^2\right)^3-3\left(2a^2b^2c^2+\sum_{cyc}\left(a^2b^2\left(\sum_{cyc}a^2-c^2\right)\right)\right)$$

$$=\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)$$

$$\therefore\sum_{cyc}a^6\overset{(2)}{=}\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)$$

$$\sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4 = \sum_{cyc}\left(a^2b^2\left(\sum_{cyc} a^2 - c^2\right)\right) \overset{(3)}{=}$$

$$\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) - 3a^2b^2c^2 \therefore (1),(2),(3) \Rightarrow m_a^2m_b^2m_c^2$$

$$= \frac{1}{64}\begin{pmatrix} -4\left(\sum_{cyc} a^2\right)^3 - 12a^2b^2c^2 + 12\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) \\ +6\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) - 18a^2b^2c^2 + 3a^2b^2c^2 \end{pmatrix}$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\left(\sum_{cyc} ab\right)^2 - 16Rrs^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\left\{\begin{matrix} -32(s^2-4Rr-r^2)^3 + 36(s^2-4Rr-r^2)(s^2+4Rr+r^2)^2 \\ -576Rrs^2(s^2-4Rr-r^2) - 432R^2r^2s^2 \end{matrix}\right\}$$

$$= \frac{1}{16}\left\{s^6 - s^4(12Rr-33r^2) - s^2(60R^2r^2+120Rr^3+33r^4) - r^3(4R+r)^3\right\}$$

$$\le \frac{R^2s^4}{4} \Leftrightarrow$$

$$s^6 - s^4(4R^2+12Rr-33r^2) - s^2(60R^2r^2+120Rr^3+33r^4) - r^3(4R+r)^3 \overset{(\bullet)}{\le} 0$$

Now, LHS of $(\bullet) \overset{\text{Gerretsen}}{\le} -s^4(8Rr-36r^2) - s^2(60R^2r^2+120Rr^3+33r^4)$
$-r^3(4R+r)^3 \overset{?}{\le} 0$
$\Leftrightarrow s^4(8R-16r) + s^2(60R^2r+120Rr^2+33r^3) + r^2(4R+r)^3 \underset{(\bullet\bullet)}{\overset{?}{\ge}} 20rs^4$

Now, LHS of $(\bullet\bullet) \underset{(a)}{\underbrace{\overset{\text{Gerretsen}}{\ge}}} s^2(16Rr-5r^2)(8R-16r)$
$+s^2(60R^2r+120Rr^2+33r^3) + r^2(4R+r)^3$ *and*
RHS of $(\bullet\bullet) \underset{(b)}{\underbrace{\overset{\text{Geretsen}}{\le}}} 20rs^2(4R^2+4Rr+3r^2)$

$(a),(b) \Rightarrow$ in order to prove $(\bullet\bullet)$, it suffices to prove :
$s^2(16Rr-5r^2)(8R-16r) + s^2(60R^2r+120Rr^2+33r^3) + r^2(4R+r)^3$
$\ge 20rs^2(4R^2+4Rr+3r^2)$
$\Leftrightarrow s^2(108R^2-256Rr+53r^2) + r(4R+r)^3 \ge 0$

$\Leftrightarrow s^2(108R^2-256Rr+80r^2)+r(4R+r)^3 \overset{(\bullet\bullet\bullet)}{\geq} 27r^2s^2$

Now, LHS of $(\bullet\bullet\bullet) \underbrace{\overset{\text{Gerretsen}}{\geq}}_{(c)} (108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3$

and RHS of $(\bullet\bullet\bullet) \underbrace{\overset{\text{Geretsen}}{\leq}}_{(d)} 27r^2(4R^2+4Rr+3r^2)$

$(c),(d) \Rightarrow$ in order to prove $(\bullet\bullet\bullet)$, it suffices to prove :

$(108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3 \geq 27r^2(4R^2+4Rr+3r^2)$

$\Leftrightarrow 224t^3-587t^2+308t-60 \geq 0 \left(\text{where } t=\frac{R}{r}\right)$

$\Leftrightarrow (t-2)\big((t-2)(224t+309)+648\big) \geq 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (\bullet\bullet\bullet) \Rightarrow (\bullet\bullet)$

$\Rightarrow (\bullet)$ is true $\Rightarrow m_a^2m_b^2m_c^2 \leq \frac{R^2s^4}{4} \Rightarrow m_am_bm_c \leq \frac{Rs^2}{2}$ (QED)