ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{a}{b+c}\cdot\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)+\frac{b}{c+a}\cdot\left(\csc\frac{C}{2}+\csc\frac{A}{2}\right)+\frac{c}{a+b}\cdot\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)\geq 6$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$\forall$ A, B, C > 0, (A + B), (B + C), (C + A) form sides of a triangle
($\because$ (A + B) + (B + C) > (C + A) and analogs) $\Rightarrow \sqrt{A+B}, \sqrt{B+C}, \sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2 =$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\rightarrow(1)$$

$$\text{Now, }\forall x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\rightarrow\text{true}\ \therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\rightarrow(2)$$

$$\text{We have : }\frac{a}{b+c}\cdot\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)+\frac{b}{c+a}\cdot\left(\csc\frac{C}{2}+\csc\frac{A}{2}\right)+\frac{c}{a+b}\cdot\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=a,y=b,z=c,A=\csc\frac{A}{2},B=\csc\frac{B}{2},C=\csc\frac{C}{2}\right)$$

$$=\frac{x}{y+z}\cdot\sqrt{B+C}^2+\frac{y}{z+x}\cdot\sqrt{C+A}^2+\frac{z}{x+y}\cdot\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F\cdot\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}\cdot\frac{\sqrt{3}}{2}=\sqrt{3}\cdot\sqrt{\sum_{cyc}\left(\csc\frac{A}{2}\cdot\csc\frac{B}{2}\right)}$$

$$\overset{A-G}{\geq}\sqrt{3}\cdot\sqrt{3\cdot\sqrt[3]{\prod_{cyc}\csc^2\frac{A}{2}}}=\sqrt{3}\cdot\sqrt{3\cdot\sqrt[3]{\frac{16R^2}{r^2}}}\overset{\text{Euler}}{\geq}\sqrt{3}\cdot\sqrt{3\cdot\sqrt[3]{64}}=6$$

$$\therefore\frac{a}{b+c}\cdot\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)+\frac{b}{c+a}\cdot\left(\csc\frac{C}{2}+\csc\frac{A}{2}\right)+\frac{c}{a+b}\cdot\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)\geq 6$$

$\forall\ \Delta$ ABC, " = " iff Δ ABC is equilateral (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Mollweide's formula, we have

$$\frac{a}{b+c}=\frac{\sin\frac{A}{2}}{\cos\frac{B-C}{2}}\geq\sin\frac{A}{2}.\ \text{(and analogs)}$$

Then

$$\sum_{cyc}\frac{a}{b+c}\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)\geq\sum_{cyc}\frac{\csc\frac{B}{2}+\csc\frac{C}{2}}{\csc\frac{A}{2}}=\sum_{cyc}\left(\frac{\csc\frac{B}{2}}{\csc\frac{C}{2}}+\frac{\csc\frac{C}{2}}{\csc\frac{B}{2}}\right)\overset{AM-GM}{\geq}\sum_{cyc}2=6.$$

Equality holds iff ΔABC is equilateral.