ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{h_a}{h_b+h_c}.\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)+\frac{h_b}{h_c+h_a}.\left(\csc\frac{C}{2}+\csc\frac{A}{2}\right)+\frac{h_c}{h_a+h_b}.\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)\ge 6$$

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$\forall$ A, B, C >0, $(A+B),(B+C),(C+A)$ form sides of a triangle $(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$

Now, $\forall x,y,z>0$, $\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\ge}}\frac{3}{4}$

Via Bergstrom, LHS of $(*)\ge\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$

$\overset{?}{\ge}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\ge}3xyz\sum_{cyc}x\to$ true $\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt{3}}{2}\to(2)$

We have : $\frac{h_a}{h_b+h_c}.\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)+\frac{h_b}{h_c+h_a}.\left(\csc\frac{C}{2}+\csc\frac{A}{2}\right)$

$+\frac{h_c}{h_a+h_b}.\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$

$\left(x=h_a,y=h_b,z=h_c,A=\csc\frac{A}{2},B=\csc\frac{B}{2},C=\csc\frac{C}{2}\right)$

$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\ge}$

$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\ge}2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}=\sqrt{3}.\sqrt{\sum_{cyc}\left(\csc\frac{A}{2}.\csc\frac{B}{2}\right)}$

$$\overset{A-G}{\ge}\sqrt{3}.\sqrt{3.\sqrt[3]{\prod_{cyc}\csc^2\frac{A}{2}}}=\sqrt{3}.\sqrt{3.\sqrt[3]{\frac{16R^2}{r^2}}}\overset{\text{Euler}}{\ge}\sqrt{3}.\sqrt{3.\sqrt[3]{64}}=6$$

$\therefore\frac{h_a}{h_b+h_c}.\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)+\frac{h_b}{h_c+h_a}.\left(\csc\frac{C}{2}+\csc\frac{A}{2}\right)+\frac{h_c}{h_a+h_b}.\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)$
$\ge 6\ \forall\ \Delta$ ABC, $''=''$ iff Δ ABC is equilateral (QED)