ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{\csc\frac{A}{2}}{\csc\frac{B}{2}+\csc\frac{C}{2}}.(b+c)+\frac{\csc\frac{B}{2}}{\csc\frac{C}{2}+\csc\frac{A}{2}}.(c+a)+\frac{\csc\frac{C}{2}}{\csc\frac{A}{2}+\csc\frac{B}{2}}.(a+b)\ge 6\sqrt{3}r$$

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$\forall$ A, B, C > 0, $(A+B),(B+C),(C+A)$ form sides of a triangle

$(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and

$$16F^2 = 2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2 =$$

$$2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$

Now, $\forall x,y,z>0$, $\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\ge}}\frac{3}{4}$

Via Bergstrom,

$$\text{LHS of }(*)\ge\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=$$

$$\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}\overset{?}{\ge}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\ge}3xyz\sum_{cyc}x\to\text{true}$$

$$\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt{3}}{2}\to(2)$$

We have : $\frac{\csc\frac{A}{2}}{\csc\frac{B}{2}+\csc\frac{C}{2}}.(b+c)+\frac{\csc\frac{B}{2}}{\csc\frac{C}{2}+\csc\frac{A}{2}}.(c+a)+\frac{\csc\frac{C}{2}}{\csc\frac{A}{2}+\csc\frac{B}{2}}.(a+b)$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=\csc\frac{A}{2},y=\csc\frac{B}{2},z=\csc\frac{C}{2},A=a,B=b,C=c\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via }(1)\text{ and }(2)}{\geq}2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}=\sqrt{3}.\sqrt{\sum_{cyc}ab}$$

$$\overset{\text{Gordon}}{\geq}\sqrt{3}.\sqrt{4\sqrt{3}rs}\overset{\text{Mitrinovic}}{\geq}6.\sqrt{3\sqrt{3}r^2}=6\sqrt{3}r$$

$$\therefore\frac{\csc\frac{A}{2}}{\csc\frac{B}{2}+\csc\frac{C}{2}}.(b+c)+\frac{\csc\frac{B}{2}}{\csc\frac{C}{2}+\csc\frac{A}{2}}.(c+a)+\frac{\csc\frac{C}{2}}{\csc\frac{A}{2}+\csc\frac{B}{2}}.(a+b)\geq 6\sqrt{3}r$$

$\forall\,\Delta\,ABC,''=''$ iff $\Delta\,ABC$ is equilateral (QED)