In any Δ ABC and $\forall$ n $\in \mathbb{N}$, the following relationships hold :

$$\text{①}\ \frac{a^n}{b^n+c^n}\cdot\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)+\frac{b^n}{c^n+a^n}\cdot\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)+\frac{c^n}{a^n+b^n}\cdot\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)\ge 3\sqrt{3}$$

$$\text{②}\ \frac{h_a^n}{h_b^n+h_c^n}\cdot\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)+\frac{h_b^n}{h_c^n+h_a^n}\cdot\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)+\frac{h_c^n}{h_a^n+h_b^n}\cdot\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)\ge 3\sqrt{3}$$

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Solution 1 by Soumava Chakraborty-Kolkata-India

$\forall$ A, B, C > 0, (A + B), (B + C), (C + A) form sides of a triangle $(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow \sqrt{A+B}, \sqrt{B+C}, \sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2 =$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2 = 2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\rightarrow(1)$$

$$\text{Now, }\forall x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\ge}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\ge\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\ge}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\ge}3xyz\sum_{cyc}x\rightarrow\text{true}\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt{3}}{2}\rightarrow(2)$$

$$\text{We have : }\frac{a^n}{b^n+c^n}\cdot\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)+\frac{b^n}{c^n+a^n}\cdot\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)$$

$$+\frac{c^n}{a^n+b^n}\cdot\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=a^n,y=b^n,z=c^n,A=\cot\frac{A}{2},B=\cot\frac{B}{2},C=\cot\frac{C}{2}\right)$$

$$=\frac{x}{y+z}\cdot\sqrt{B+C}^2+\frac{y}{z+x}\cdot\sqrt{C+A}^2+\frac{z}{x+y}\cdot\sqrt{A+B}^2\overset{\text{Oppenheim}}{\ge}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\ge}2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}=\sqrt{3}.\sqrt{\sum_{cyc}\left(\cot\frac{A}{2}.\cot\frac{B}{2}\right)}$$

$$=\sqrt{3}.\sqrt{s^2\sum_{cyc}\frac{1}{r_ar_b}}=\sqrt{3}.\sqrt{s^2.\frac{4R+r}{rs^2}}\overset{\text{Euler}}{\ge}\sqrt{3}.\sqrt{\frac{8r+r}{r}}=3\sqrt{3}$$

$$\therefore\frac{a^n}{b^n+c^n}\cdot\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)+\frac{b^n}{c^n+a^n}\cdot\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)+\frac{c^n}{a^n+b^n}\cdot\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)$$

$\ge 3\sqrt{3}\ \forall\ \Delta$ ABC, " = " iff Δ ABC is equilateral (QED)

$$\text{Again, we have : }\frac{h_a^n}{h_b^n+h_c^n}\cdot\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)+\frac{h_b^n}{h_c^n+h_a^n}\cdot\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)$$

$$+\frac{h_c^n}{h_a^n+h_b^n}\cdot\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=h_a^n, y=h_b^n, z=h_c^n, A=\cot\frac{A}{2}, B=\cot\frac{B}{2}, C=\cot\frac{C}{2}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}=\sqrt{3}.\sqrt{\sum_{cyc}\left(\cot\frac{A}{2}.\cot\frac{B}{2}\right)}$$

$$=\sqrt{3}.\sqrt{s^2\sum_{cyc}\frac{1}{r_a r_b}}=\sqrt{3}.\sqrt{s^2.\frac{4R+r}{rs^2}}\overset{\text{Euler}}{\geq}\sqrt{3}.\sqrt{\frac{8r+r}{r}}=3\sqrt{3}$$

$$\therefore\frac{h_a^n}{h_b^n+h_c^n}.\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)+\frac{h_b^n}{h_c^n+h_a^n}.\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)+\frac{h_c^n}{h_a^n+h_b^n}.\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)\geq 3\sqrt{3}\ \forall\ \Delta\ ABC,$$

$$''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

Solution 2 by Tapas Das-India

Walter Janous inequality : ***for a, b, c and x, y, z be positive real numbers*** :

$$\frac{x}{y+z}(b+c)+\frac{y}{z+x}(c+a)+\frac{z}{x+y}(a+b)\geq\sqrt{3(ab+bc+ca)}$$

Using this solution of (1)&(2)

$$1)\ \frac{a^n}{b^n+c^n}\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)+\frac{b^n}{c^n+a^n}\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)+\frac{c^n}{a^n+b^n}\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)\geq$$

$$\geq\sqrt{3\sum\cot\frac{A}{2}\cot\frac{B}{2}}=\sqrt{\frac{3s^2(r_a+r_b+r_c)}{r_ar_br_c}}=\sqrt{\frac{3(4R+r)}{r}}\overset{Euler}{\geq}\left(\frac{27r}{r}\right)^{\frac{1}{2}}=3\sqrt{3}$$

$$2)\ \frac{h_a^n}{h_b^n+h_c^n}\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)+\frac{h_b^n}{h_c^n+h_a^n}\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)+\frac{h_c^n}{h_a^n+h_b^n}\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)\geq$$

$$\geq\sqrt{3\sum\cot\frac{A}{2}\cot\frac{B}{2}}=\sqrt{\frac{3s^2(r_a+r_b+r_c)}{r_ar_br_c}}=\sqrt{\frac{3(4R+r)}{r}}\overset{Euler}{\geq}\left(\frac{27r}{r}\right)^{\frac{1}{2}}=3\sqrt{3}$$