ROMANIAN MATHEMATICAL MAGAZINE

In any acute Δ ABC, the following relationship holds :

$$\frac{a}{b+c}.(\sec B+\sec C)+\frac{b}{c+a}.(\sec C+\sec A)+\frac{c}{a+b}.(\sec A+\sec B)\ge 6$$

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$\forall$ A, B, C > 0, (A + B), (B + C), (C + A) form sides of a triangle $(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$

$$\text{Now, }\forall x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt3}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\ge}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\ge\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\ge}\frac34\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\ge}3xyz\sum_{cyc}x\to\text{true}\ \therefore\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt3}{2}\to(2)$$

$$\text{We have : }\frac{a}{b+c}.(\sec B+\sec C)+\frac{b}{c+a}.(\sec C+\sec A)+\frac{c}{a+b}.(\sec A+\sec B)$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$(x=a, y=b, z=c, A=\sec A, B=\sec B, C=\sec C)$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\ge}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\ge}2\sqrt{\sum_{cyc}AB}.\frac{\sqrt3}{2}=\sqrt3.\sqrt{\sum_{cyc}\sec A\sec B}$$

$$=\sqrt{3\left(\prod_{cyc}\sec A\right)\left(\sum_{cyc}\cos A\right)}=\sqrt{\frac{12R^2(R+r)}{R(s^2-4R^2-4Rr-r^2)}}\overset{\text{Gerretsen}}{\ge}$$

$$\sqrt{\frac{12R(R+r)}{4R^2+4Rr+3r^2-4R^2-4Rr-r^2}}=\sqrt{\frac{6R(R+r)}{r^2}}\overset{\text{Euler}}{\ge}\sqrt{\frac{12r(3r)}{r^2}}=6$$

$$\therefore\frac{a}{b+c}.(\sec B+\sec C)+\frac{b}{c+a}.(\sec C+\sec A)+\frac{c}{a+b}.(\sec A+\sec B)\ge 6$$

$\forall$ acute Δ ABC, '' = '' iff Δ ABC is equilateral (QED)