ROMANIAN MATHEMATICAL MAGAZINE

In any acute Δ ABC, the following relationship holds :

$$\frac{\sec A}{\sec B+\sec C}.(b+c)+\frac{\sec B}{\sec C+\sec A}.(c+a)+\frac{\sec C}{\sec A+\sec B}.(a+b)\geq 6\sqrt{3}r$$

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$\forall$ A, B, C > 0, (A + B), (B + C), (C + A) form sides of a triangle
$(\because (A+B)+(B+C)>(C+A)$ and analogs$) \Rightarrow \sqrt{A+B}, \sqrt{B+C}, \sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2 =$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2 = 2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$= 6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB \Rightarrow 4F = 2\sqrt{\sum_{cyc}AB} \rightarrow (1)$$

Now, $\forall\, x, y, z > 0$, $\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)} \underset{(*)}{\overset{?}{\geq}} \frac{3}{4}$

Via Bergstrom, LHS of $(*) \geq \frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)} = \frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$

$$\overset{?}{\geq}\frac{3}{4} \Leftrightarrow \left(\sum_{cyc}xy\right)^2 \overset{?}{\geq} 3xyz\sum_{cyc}x \rightarrow \text{true} \therefore \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \rightarrow (2)$$

We have : $\frac{\sec A}{\sec B+\sec C}.(b+c)+\frac{\sec B}{\sec C+\sec A}.(c+a)+\frac{\sec C}{\sec A+\sec B}.(a+b)$

$$= \frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$(x = \sec A, y = \sec B, z = \sec C, A = a, B = b, C = c)$

$$= \frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2 \overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2} = \sqrt{3}.\sqrt{\sum_{cyc}ab}$$

$$\overset{\text{Gordon}}{\geq} \sqrt{3}.\sqrt{4\sqrt{3}rs} \overset{\text{Mitrinovic}}{\geq} \sqrt{3}.\sqrt{4\sqrt{3}.3\sqrt{3}r^2} = 6\sqrt{3}r$$

$$\therefore \frac{\sec A}{\sec B+\sec C}.(b+c)+\frac{\sec B}{\sec C+\sec A}.(c+a)+\frac{\sec C}{\sec A+\sec B}.(a+b)\geq 6\sqrt{3}r$$

$\forall\, \Delta$ ABC, " = " iff Δ ABC is equilateral (QED)