ROMANIAN MATHEMATICAL MAGAZINE

In any $\Delta\ ABC$, the following relationship holds :

① $\dfrac{a^n}{b^n+c^n}\cdot\left(\csc\dfrac{B}{2}+\csc\dfrac{C}{2}\right)+\dfrac{b^n}{c^n+a^n}\cdot\left(\csc\dfrac{C}{2}+\csc\dfrac{A}{2}\right)+\dfrac{c^n}{a^n+b^n}\cdot\left(\csc\dfrac{A}{2}+\csc\dfrac{B}{2}\right)\ge 6$

② $\dfrac{h_a^n}{h_b^n+h_c^n}\cdot\left(\csc\dfrac{B}{2}+\csc\dfrac{C}{2}\right)+\dfrac{h_b^n}{h_c^n+h_a^n}\cdot\left(\csc\dfrac{C}{2}+\csc\dfrac{A}{2}\right)+\dfrac{h_c^n}{h_a^n+h_b^n}\cdot\left(\csc\dfrac{A}{2}+\csc\dfrac{B}{2}\right)\ge 6$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall\ A,B,C>0,(A+B),(B+C),(C+A)$ form sides of a triangle

$(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow\sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and

$$16F^2=2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=$$

$$=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB=$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$

Now, $\forall\ x,y,z>0,\ \sqrt{\sum_{cyc}\dfrac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\dfrac{\sqrt3}{2}\Leftrightarrow\sum_{cyc}\dfrac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\ge}}\dfrac34$

Via Bergstrom, LHS of $(*)\ge\dfrac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\dfrac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$

$\overset{?}{\ge}\dfrac34\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\ge}3xyz\sum_{cyc}x\to$ true $\therefore\sqrt{\sum_{cyc}\dfrac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\dfrac{\sqrt3}{2}\to(2)$

We have : $\dfrac{x}{y+z}\cdot\left(\csc\dfrac{B}{2}+\csc\dfrac{C}{2}\right)+\dfrac{y}{z+x}\cdot\left(\csc\dfrac{C}{2}+\csc\dfrac{A}{2}\right)+\dfrac{z}{x+y}\cdot\left(\csc\dfrac{A}{2}+\csc\dfrac{B}{2}\right)$

$=\dfrac{x}{y+z}(B+C)+\dfrac{y}{z+x}(C+A)+\dfrac{z}{x+y}(A+B)\ \left(A=\csc\dfrac{A}{2},B=\csc\dfrac{B}{2},C=\csc\dfrac{C}{2}\right)$

$$=\frac{x}{y+z}\cdot\sqrt{B+C}^2+\frac{y}{z+x}\cdot\sqrt{C+A}^2+\frac{z}{x+y}\cdot\sqrt{A+B}^2\overset{\text{Oppenheim}}{\ge}$$

$$4F\cdot\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\ge}2\sqrt{\sum_{cyc}AB}\cdot\frac{\sqrt3}{2}=\sqrt3\cdot\sqrt{\sum_{cyc}\csc\frac{A}{2}\csc\frac{B}{2}}$$

$$\overset{A-G}{\geq} 3.\sqrt[6]{\csc^2\frac{A}{2}\csc^2\frac{B}{2}\csc^2\frac{C}{2}} = 3.\sqrt[6]{\frac{16R^2}{r^2}} \overset{Euler}{\geq} 3.\sqrt[6]{\frac{64r^2}{r^2}} = 6$$

$$\therefore \forall x, y, z > 0, \frac{x}{y+z}.\left(\csc\frac{B}{2} + \csc\frac{C}{2}\right) + \frac{y}{z+x}.\left(\csc\frac{C}{2} + \csc\frac{A}{2}\right) + \frac{z}{x+y}.\left(\csc\frac{A}{2} + \csc\frac{B}{2}\right) \geq 6$$

$\therefore$ **choosing** $x = a^n, y = b^n, z = c^n$ *and* $x = h_a^n, y = h_b^n, z = h_c^n$ **separately, we arrive at:**

$$\frac{a^n}{b^n + c^n}.\left(\csc\frac{B}{2} + \csc\frac{C}{2}\right) + \frac{b^n}{c^n + a^n}.\left(\csc\frac{C}{2} + \csc\frac{A}{2}\right) + \frac{c^n}{a^n + b^n}.\left(\csc\frac{A}{2} + \csc\frac{B}{2}\right) \geq 6$$

and

$$\frac{h_a^n}{h_b^n + h_c^n}.\left(\csc\frac{B}{2} + \csc\frac{C}{2}\right) + \frac{h_b^n}{h_c^n + h_a^n}.\left(\csc\frac{C}{2} + \csc\frac{A}{2}\right) + \frac{h_c^n}{h_a^n + h_b^n}.\left(\csc\frac{A}{2} + \csc\frac{B}{2}\right) \geq 6$$

respectively, $\forall \Delta ABC$, $'' = ''$ **iff** ΔABC **is equilateral (QED)**