ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^3+\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^3+\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^3 \geq \frac{3.2^9.r^6}{3(9R^3-64r^3)^2-128r^6}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^3+\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^3+\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^3 \overset{A-G}{\geq}$$

$$3\left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right) \overset{A-G}{\geq} 3.8.\sqrt{\frac{m_a}{w_b}.\frac{w_c}{h_a}.\frac{m_b}{w_c}.\frac{w_a}{h_b}.\frac{m_c}{w_a}.\frac{w_b}{h_c}} \geq 3.8$$

$$\overset{?}{\geq} \frac{3.2^9.r^6}{3(9R^3-64r^3)^2-128r^6} \Leftrightarrow (9R^3-64r^3)^2 \overset{?}{\geq} 64r^6 \Leftrightarrow 9R^3-64r^3 \overset{?}{\geq} 8r^3$$

$$\Leftrightarrow R^3 \overset{?}{\geq} 8r^3 \to \text{true} \because R \overset{Euler}{\geq} 2r \therefore \left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^3+\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^3+\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^3$$

$$\geq \frac{3.2^9.r^6}{3(9R^3-64r^3)^2-128r^6} \forall \Delta \text{ ABC}, ''='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$