

# ROMANIAN MATHEMATICAL MAGAZINE

**In any non – right  $\Delta ABC$ , the following relationship holds :**

$$\frac{h_a}{h_b + h_c} \cdot (\sec B + \sec C) + \frac{h_b}{h_c + h_a} \cdot (\sec C + \sec A) + \frac{h_c}{h_a + h_b} \cdot (\sec A + \sec B) \geq 6$$

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$\forall A, B, C > 0$ ,  $(A + B)$ ,  $(B + C)$ ,  $(C + A)$  form sides of a triangle

( $\because (A + B) + (B + C) > (C + A)$  and analogs)  $\Rightarrow \sqrt{A + B}$ ,  $\sqrt{B + C}$ ,  $\sqrt{C + A}$  form sides of a triangle with area  $F$  (say) and  $16F^2 =$

$$\begin{aligned} 2 \sum_{\text{cyc}} (A + B)(B + C) - \sum_{\text{cyc}} (A + B)^2 &= 2 \sum_{\text{cyc}} \left( \sum_{\text{cyc}} AB + B^2 \right) - 2 \sum_{\text{cyc}} A^2 - 2 \sum_{\text{cyc}} AB \\ &= 6 \sum_{\text{cyc}} AB + 2 \sum_{\text{cyc}} A^2 - 2 \sum_{\text{cyc}} A^2 - 2 \sum_{\text{cyc}} AB \Rightarrow 4F = 2 \sqrt{\sum_{\text{cyc}} AB} \rightarrow (1) \end{aligned}$$

$$\text{Now, } \forall x, y, z > 0, \sqrt{\sum_{\text{cyc}} \frac{xy}{(y+z)(z+x)}} \stackrel{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{\text{cyc}} \frac{x^2 y^2}{xy(y+z)(z+x)} \stackrel{?}{\geq} \frac{3}{4} \quad (*)$$

$$\text{Via Bergstrom, LHS of } (*) \geq \frac{(\sum_{\text{cyc}} xy)^2}{\sum_{\text{cyc}} (xy(\sum_{\text{cyc}} xy + z^2))} = \frac{(\sum_{\text{cyc}} xy)^2}{(\sum_{\text{cyc}} xy)^2 + xyz \sum_{\text{cyc}} x}$$

$$\stackrel{?}{\geq} \frac{3}{4} \Leftrightarrow \left( \sum_{\text{cyc}} xy \right)^2 \stackrel{?}{\geq} 3xyz \sum_{\text{cyc}} x \rightarrow \text{true} \therefore \sqrt{\sum_{\text{cyc}} \frac{xy}{(y+z)(z+x)}} \stackrel{?}{\geq} \frac{\sqrt{3}}{2} \rightarrow (2)$$

$$\text{We have : } \frac{h_a}{h_b + h_c} \cdot (\sec B + \sec C) + \frac{h_b}{h_c + h_a} \cdot (\sec C + \sec A)$$

$$+ \frac{h_c}{h_a + h_b} \cdot (\sec A + \sec B) = \frac{x}{y+z} (B + C) + \frac{y}{z+x} (C + A) + \frac{z}{x+y} (A + B)$$

$$(x = h_a, y = h_b, z = h_c, A = \sec A, B = \sec B, C = \sec C)$$

$$= \frac{x}{y+z} \cdot \sqrt{B+C}^2 + \frac{y}{z+x} \cdot \sqrt{C+A}^2 + \frac{z}{x+y} \cdot \sqrt{A+B}^2 \stackrel{\text{Oppenheim}}{\geq}$$

$$4F \cdot \sqrt{\sum_{\text{cyc}} \frac{xy}{(y+z)(z+x)}} \stackrel{\text{via (1) and (2)}}{\geq} 2 \sqrt{\sum_{\text{cyc}} AB} \cdot \frac{\sqrt{3}}{2} = \sqrt{3} \cdot \sqrt{\sum_{\text{cyc}} \sec A \sec B}$$

$$= \sqrt{3(\sec A \sec B \sec C) \sum_{\text{cyc}} \cos A} = \sqrt{3 \left( \frac{4R^2}{s^2 - 4R^2 - 4Rr - r^2} \right) \left( \frac{R+r}{R} \right)} \stackrel{\text{Gerretsen}}{\geq}$$

$$\sqrt{3 \left( \frac{4R^2}{4R^2 + 4Rr + 3r^2 - 4R^2 - 4Rr - r^2} \right) \left( \frac{R+r}{R} \right)} = \sqrt{\frac{6R(R+r)}{r^2}} \stackrel{\text{Euler}}{\geq} \sqrt{\frac{6(2r)(3r)}{r^2}}$$

$$= 6 \therefore \frac{h_a}{h_b + h_c} \cdot (\sec B + \sec C) + \frac{h_b}{h_c + h_a} \cdot (\sec C + \sec A)$$

$$+ \frac{h_c}{h_a + h_b} \cdot (\sec A + \sec B) \geq 6 \forall \text{ non – right } \Delta ABC,$$

" = " iff  $\Delta ABC$  is equilateral (QED)