

ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{a}{b \left(\sin \frac{A}{2} + \sin \frac{B}{2} \right) + c \sin \frac{C}{2}} + \frac{b}{c \left(\sin \frac{A}{2} + \sin \frac{B}{2} \right) + a \sin \frac{C}{2}} + \frac{c}{a \left(\sin \frac{A}{2} + \sin \frac{B}{2} \right) + b \sin \frac{C}{2}} \geq 2$$

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$$\text{Note: } \sum \sin \frac{A}{2} \stackrel{\text{Jensen}}{\leq} 3 \sin \left(\frac{A+B+C}{6} \right) = 3 \sin \frac{\pi}{6} = \frac{3}{2} \quad (1)$$

$$\frac{a}{b \left(\sin \frac{A}{2} + \sin \frac{B}{2} \right) + c \sin \frac{C}{2}} + \frac{b}{c \left(\sin \frac{A}{2} + \sin \frac{B}{2} \right) + a \sin \frac{C}{2}} + \frac{c}{a \left(\sin \frac{A}{2} + \sin \frac{B}{2} \right) + b \sin \frac{C}{2}} \geq 2$$

$$\begin{aligned} \text{or } \sum \frac{a}{b \left(\sin \frac{A}{2} + \sin \frac{B}{2} \right) + c \sin \frac{C}{2}} &= \sum \frac{a^2}{ab \left(\sin \frac{A}{2} + \sin \frac{B}{2} \right) + ca \sin \frac{C}{2}} \stackrel{\text{Bergstrom}}{\geq} \\ &\geq \frac{(a+b+c)^2}{(\sum ab) (\sum \sin \frac{A}{2})} \stackrel{3 \sum ab \leq (\sum a)^2}{\geq} \frac{(\sum a)^2}{\left(\frac{(\sum a)^2}{3} \right) \left(\frac{3}{2} \right)} \text{ (using (1))} = 2 \end{aligned}$$

Equality for $a = b = c$