ROMANIAN MATHEMATICAL MAGAZINE

In acute ΔABC holds:

$$h_a(\sec B+\sec C)+h_b(\sec C+\sec A)+h_c(\sec A+\sec A)\geq 36r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$let\ f(x)=\sec x\,, x\in\left(0,\frac{\pi}{2}\right), f''(x)=\sec x\tan^2 x+\sec^3 x>0\,,$$

$$so\ f\ is\ convex\in\left(0,\frac{\pi}{2}\right).\ Using\ Jensen\ inequality:$$

$$f(A)+f(B)+f(C)\geq 3f\left(\frac{A+B+C}{3}\right)=3f\left(\frac{\pi}{3}\right)\ or,\sum\sec A\geq 3\sec\frac{\pi}{3}=6\ \ (1)$$

$$,\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r},\qquad \sum h_a\overset{AM-GM}{\geq}3\sqrt[3]{h_ah_bh_c}\overset{Gm-Hm}{\geq}3.\frac{3}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}=9r\ \ (2)$$

$$WLOG\ a\geq b\geq c\ then:$$

$$h_a\leq h_b\leq h_c\ and\ \sec A\geq\sec B\geq\sec C\ and$$

$$\sec A+\sec B\geq\sec A+\sec C\geq\sec C+\sec B$$

$$h_a(\sec B+\sec C)+h_b(\sec C+\sec A)+h_c(\sec A+\sec A)\overset{Chebyshev}{\geq}$$

$$\geq\frac{1}{3}\left(\sum h_a\right)\left(\sum\sec A+\sec B\right)=\frac{1}{3}\left(\sum h_a\right)\left(2\sum\sec A\right)\overset{(1)\&(2)}{\geq}$$

$$\geq\frac{1}{3}.9r.2.6=36r$$

$$Equality\ for\ a=b=c$$