ROMANIAN MATHEMATICAL MAGAZINE

In any acute ΔABC the following relationship holds:

$$a(sec\,B+sec\,C)+b\,(sec\,C+sec\,A)+c\,(sec\,B+sec\,C)\ge 24\sqrt{3}r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$\sum \cos A = 1+\frac{r}{R}\overset{\text{Euler}}{\le}\frac{3}{2},\prod \cos A \overset{\text{AM-GM}}{\le}\left(\frac{\sum \cos A}{3}\right)^3 \le \left(\frac{3}{2}\cdot\frac{1}{3}\right)^3=\frac{1}{8}$$

$$\text{so} \prod \sec A \ge 8 \quad (1)$$

$$a(se\,c\,B+se\,c\,C)+b\,(se\,c\,C+se\,c\,A)+c\,(se\,c\,B+se\,c\,C)=$$

$$=\sum a(se\,c\,B+se\,c\,C)\overset{\text{AM-GM}}{\ge}$$

$$2\sum a\sqrt{\sec B\sec C}\overset{AM-GM}{\ge} 6\sqrt[3]{abc\sec B\sec C\sec A}\overset{(1)}{\ge}$$

$$\ge 6\sqrt[3]{4Rrs.8}\overset{Euler\,\&\,Mritrinovic}{\ge} 6\left(4.2r.r.3\sqrt{3}r.8\right)^{\frac{1}{3}}=24\sqrt{3}\,r$$

$$Equality\ holds\ \ for\ a=b=c$$