ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{h_a^n}{h_b^n+h_c^n}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+\frac{h_b^n}{h_c^n+h_a^n}\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+\frac{h_c^n}{h_a^n+h_b^n}\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\geq\sqrt{3},\qquad n\in\mathbb{N}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$WLOG\ a\geq b\geq c\text{ , }then$$

$$h_a\leq h_b\leq h_c\ and\ \ h_a+h_b\leq h_a+h_c\leq h_b+h_c\ ,$$

$$\frac{h_a^n}{h_b^n+h_c^n}\leq\frac{h_b^n}{h_c^n+h_a^n}\leq\frac{h_c^n}{h_a^n+h_b^n}$$

$$and\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)\leq\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)\leq\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)$$

$$\frac{h_a^n}{h_b^n+h_c^n}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+\frac{h_b^n}{h_c^n+h_a^n}\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+\frac{h_c^n}{h_a^n+h_b^n}\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\geq$$

$$\overset{Chebyshev}{\geq}\frac{1}{3}\cdot\sum\frac{h_a^n}{h_b^n+h_c^n}\sum\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)\overset{Nesbitt}{\geq}$$

$$\geq\frac{1}{3}\cdot\frac{3}{2}\cdot\frac{2(4R+r)}{s}\geq\sqrt{3}\ \left(since\frac{4R+r}{s}\geq\sqrt{3}\right)$$

Equality for a=b=c