

ROMANIAN MATHEMATICAL MAGAZINE

In $\triangle ABC$ the following relationship holds:

$$\frac{a}{b\left(\cos\frac{A}{2} + \cos\frac{B}{2}\right) + c\cos\frac{C}{2}} + \frac{b}{c\left(\cos\frac{A}{2} + \cos\frac{B}{2}\right) + a\cos\frac{C}{2}} + \frac{c}{a\left(\cos\frac{A}{2} + \cos\frac{B}{2}\right) + a\cos\frac{C}{2}} \geq \frac{2}{\sqrt{3}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$\begin{aligned} \sum \cos\frac{A}{2} &= \sqrt{\left(\sum \cos\frac{A}{2}\right)^2} \leq \sqrt{3 \sum \left(\cos^2\frac{A}{2}\right)} = \sqrt{3\left(2 + \frac{r}{2R}\right)} \stackrel{\text{Euler}}{\leq} \sqrt{3\left(2 + \frac{1}{4}\right)} = \frac{3\sqrt{3}}{2} \quad (1) \\ &\frac{a}{b\left(\cos\frac{A}{2} + \cos\frac{B}{2}\right) + c\cos\frac{C}{2}} + \frac{b}{c\left(\cos\frac{A}{2} + \cos\frac{B}{2}\right) + a\cos\frac{C}{2}} + \frac{c}{a\left(\cos\frac{A}{2} + \cos\frac{B}{2}\right) + a\cos\frac{C}{2}} = \\ &= \sum \frac{a}{b\left(\cos\frac{A}{2} + \cos\frac{B}{2}\right) + c\cos\frac{C}{2}} = \sum \frac{a^2}{ba\left(\cos\frac{A}{2} + \cos\frac{B}{2}\right) + ca\cos\frac{C}{2}} \stackrel{\text{Bergstrom}}{\geq} \\ &\geq \frac{(\sum a)^2}{(\sum ab)\left(\sum \cos\frac{A}{2}\right)} \stackrel{(1)}{\geq} \frac{(\sum a)^2}{\frac{(\sum a)^2}{3}\left(\frac{3\sqrt{3}}{2}\right)} = \frac{2}{\sqrt{3}} \end{aligned}$$

Equality holds for $a = b = c$