ROMANIAN MATHEMATICAL MAGAZINE

In any non − right Δ ABC, the following relationships hold :

① $\frac{a^n}{b^n+c^n}(\sec B+\sec C)+\frac{b^n}{c^n+a^n}(\sec C+\sec A)+\frac{c^n}{a^n+b^n}(\sec A+\sec B)\ge 6$

② $\frac{h_a^n}{h_b^n+h_c^n}(\sec B+\sec C)+\frac{h_b^n}{h_c^n+h_a^n}(\sec C+\sec A)+\frac{h_c^n}{h_a^n+h_b^n}(\sec A+\sec B)\ge 6$

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$\forall$ A, B, C > 0, (A + B), (B + C), (C + A) form sides of a triangle ($\because$ (A + B) + (B + C) > (C + A) and $analogs$) $\Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with $area$ F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$

$$\text{Now, }\forall x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt3}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\ge}}\frac34$$

$$\text{Via Bergstrom, LHS of }(*)\ge\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\ge}\frac34\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\ge}3xyz\sum_{cyc}x\to\text{true}\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt3}{2}\to(2)$$

$$\text{We have : }\frac{x}{y+z}.(\sec B+\sec C)+\frac{y}{z+x}.(\sec C+\sec A)+\frac{z}{x+y}.(\sec A+\sec B)$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)\ (A=\sec A,B=\sec B,C=\sec C)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\ge}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\ge}2\sqrt{\sum_{cyc}AB}.\frac{\sqrt3}{2}=\sqrt3.\sqrt{\sum_{cyc}\sec A\sec B}$$

$$=\sqrt{3(\sec A\sec B\sec C)\sum_{cyc}\cos A}=\sqrt{3\left(\frac{4R^2}{s^2-4R^2-4Rr-r^2}\right)\left(\frac{R+r}{R}\right)}$$

$$\overset{\text{Gerretsen}}{\geq} \sqrt{3\left(\frac{4R^2}{4R^2+4Rr+3r^2-4R^2-4Rr-r^2}\right)\left(\frac{R+r}{R}\right)} = \sqrt{\frac{6R(R+r)}{r^2}}$$

$$\overset{\text{Euler}}{\geq} \sqrt{\frac{6(2r)(3r)}{r^2}} = 6 \therefore \frac{x}{y+z}.(\sec B + \sec C) +$$

$\frac{y}{z+x}.(\sec C + \sec A) + \frac{z}{x+y}.(\sec A + \sec B) \geq 6$ and choosing $x = a^n, y = b^n$, $z = c^n$ and $x = h_a^n, y = h_b^n, z = h_c^n$ respectively, we get :

$$① \frac{a^n}{b^n+c^n}(\sec B + \sec C) + \frac{b^n}{c^n+a^n}(\sec C + \sec A) + \frac{c^n}{a^n+b^n}(\sec A + \sec B) \geq 6$$

$$② \frac{h_a^n}{h_b^n+h_c^n}(\sec B + \sec C) + \frac{h_b^n}{h_c^n+h_a^n}(\sec C + \sec A) + \frac{h_c^n}{h_a^n+h_b^n}(\sec A + \sec B) \geq 6$$

$\forall$ non − right Δ ABC, " = " iff Δ ABC is equilateral (QED)