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In $\Delta ABC$ the following relationship holds:

$$\frac{n_a^2 n_b^2}{\sin\frac{B}{2}\sin\frac{C}{2}}+\frac{n_b^2 n_c^2}{\sin\frac{C}{2}\sin\frac{A}{2}}+\frac{n_c^2 n_a^2}{\sin\frac{A}{2}\sin\frac{B}{2}}\geq 972r^4$$

*Proposed by Zaza Mzhavanadze-Georgia*

*Solution by Tapas Das-India*

$$h_a\leq n_a, h_b\leq n_b, h_c\leq n_c \ and \ \sum\frac{1}{h_a}=\frac{1}{r}, \sqrt[3]{h_ah_bh_c}\overset{GM\geq HM}{\geq}\frac{3}{\sum\frac{1}{h_a}}=3r(1)$$

$$And \sum\sin\frac{B}{2}\sin\frac{C}{2}\leq\frac{\left(\sum\sin\frac{B}{2}\right)^2}{3}\overset{Jensen}{\leq}\frac{\left(3\sin\frac{A+B+C}{6}\right)^2}{3}=\frac{\left(3.\frac{1}{2}\right)^2}{3}=\frac{3}{4}(2)$$

$$\frac{n_a^2 n_b^2}{\sin\frac{B}{2}\sin\frac{C}{2}}+\frac{n_b^2 n_c^2}{\sin\frac{C}{2}\sin\frac{A}{2}}+\frac{n_c^2 n_a^2}{\sin\frac{A}{2}\sin\frac{B}{2}}\geq\sum\frac{h_a^2h_b^2}{\sin\frac{B}{2}\sin\frac{C}{2}}\overset{Bergstrom}{\geq}$$

$$\frac{\left(\sum h_ah_b\right)^2}{\sum\sin\frac{B}{2}\sin\frac{C}{2}}\overset{AM-GM\,\&(2)}{\geq}\frac{9(h_ah_bh_c)^{\frac{4}{3}}}{\frac{3}{4}}\overset{(1)}{\geq}12.(3r)^4=972r^4$$

Equality holds for $a=b=c$.