ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{\left(r_a^3+r_b^3\right)^3}{\sin^2 A}+\frac{\left(r_b^3+r_c^3\right)^3}{\sin^2 B}+\frac{\left(r_c^3+r_a^3\right)^3}{\sin^2 C}\geq 32(3r)^9$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$\sum r_a^3 \overset{AM-GM}{\geq} 3r_ar_br_c=3s^2r \overset{Mitrinovic}{\geq} 3.27r^3=81r^3 \ (1)$$

$$\sum \sin A \overset{Jensen}{\leq} 3\sin\frac{A+B+C}{3}=3\sin\frac{\pi}{3}=\frac{3\sqrt{3}}{2} \ (2)$$

$$\frac{\left(r_a^3+r_b^3\right)^3}{\sin^2 A}+\frac{\left(r_b^3+r_c^3\right)^3}{\sin^2 B}+\frac{\left(r_c^3+r_a^3\right)^3}{\sin^2 C}=\sum\frac{\left(r_a^3+r_b^3\right)^3}{\sin^2 A}\overset{Radon}{\geq}$$

$$\geq\frac{\left(2\sum r_a^3\right)^3}{\left(\sum\sin A\right)^2}\overset{(1)\&(2)}{\geq} 8\frac{\left(81\,r^3\right)^3}{\left(\frac{3\sqrt{3}}{2}\right)^2}=\frac{32r^9 3^{12}}{3^3}=32(3r)^9$$

Equality holds for $a=b=c$