UNIVERSITY OF IBADAN
DEPARTMENT OF MATHEMATICS
Akerele Olofin Segun

Given $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ and $\cosh(\omega t) = \phi(t) - \int_0^t \phi(\tau)\sinh(t-\tau)\ d\tau$ where $\omega \in \mathbb{R}$
Find an expression for,

$$\int_0^1 \phi(t)\ dt$$

Solution

$$\cosh(\omega t) = \phi(t) - \int_0^t \phi(\tau)\sinh(t-\tau)\ d\tau \tag{1}$$

By taking the laplace transform of [1], then it follows;

$$\frac{s}{s^2-\omega^2} = \Phi(s) - \mathcal{L}_s\left\{\int_0^t \phi(\tau)\sinh(t-\tau)d\tau\right\}$$

Since, $\int_0^t \phi(\tau)\sinh(t-\tau)\ d\tau = \phi(t) * \sinh(t)$ (Convolution)
Then,

$$\frac{s}{s^2-\omega^2} = \Phi(s) - \mathcal{L}_s\left\{\int_0^t \phi(\tau)\sinh(t-\tau)d\tau\right\} = \Phi(s) - \mathcal{L}_s\{\phi(t)\}\mathcal{L}_s\{\sinh(t)\} = \Phi(s) - \Phi(s)\left[\frac{1}{s^2-1}\right]$$

Thus,

$$\frac{s}{s^2-\omega^2} = \Phi(s)\left[1 - \frac{1}{s^2-1}\right], \Phi(s) = \frac{s(s^2-1)}{(s^2-\omega^2)(s^2-2)}$$

Then we have,

$$\boxed{\phi(t) = \mathcal{L}^{-1}\left\{\frac{s(s^2-1)}{(s^2-\omega^2)(s^2-2)}\right\} = \frac{\omega^2}{\omega^2-2}\cosh(\omega t) - \frac{1}{\omega^2-2}\cosh(\omega t) - \frac{1}{\omega^2-2}\cosh(\sqrt{2}t)}$$

Thus,

$$\int_0^1 \phi(t)\ dt = \frac{1}{\omega^2-2}\int_0^1 \omega^2\cosh(\omega t) - \cosh(\omega t) - \cosh(\sqrt{2}t)\ dt = \frac{1}{\omega^2-2}\left[\frac{(\omega^2-1)\sinh(\omega)}{\omega} - \frac{\sinh(\sqrt{2})}{\sqrt{2}}\right]$$

. Hence we conclude,

$$\boxed{\int_0^1 \phi(t)\ dt = \frac{1}{\omega^2-2}\left[\frac{(\omega^2-1)\sinh(\omega)}{\omega} - \frac{\sinh(\sqrt{2})}{\sqrt{2}}\right], where\ \omega \neq 0, \sqrt{2}}$$