ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$I=\int_0^1 \frac{x\ln(x)\{Li_2(x)-Li_2(-x)\}}{1+x^2}dx$$

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$$I=\int_0^1 \frac{x\ln(x)\{Li_2(x)-Li_2(-x)\}}{1+x^2}dx=\int_0^1 \frac{x\ln(x)\{2Li_2(x)-[Li_2(x)+Li_2(-x)]\}}{1+x^2}dx=$$

$$=\int_0^1 \frac{x\ln(x)\{2Li_2(x)-\frac{1}{2}Li_2(x^2)\}}{1+x^2}dx=2\int_0^1 \frac{x\ln(x)Li_2(x)}{1+x^2}dx-\frac{1}{2}\int_0^1 \frac{x\ln(x)Li_2(x^2)}{1+x^2}dx$$

$in\ second\ integral\ \{x^2=u,\ 2xdx=du\}$

$$=2\int_0^1 \frac{x\ln(x)Li_2(x)}{1+x^2}dx-\frac{1}{8}\int_0^1 \frac{\ln(u)Li_2(u)}{1+u}du=2A-\frac{1}{8}B$$

Where

$$A=\int_0^1 \frac{x\ln(x)Li_2(x)}{1+x^2}dx,\qquad B=\int_0^1 \frac{\ln(x)Li_2(x)}{1+x}dx$$

$$A=\int_0^1 \frac{x\ln(x)Li_2(x)}{1+x^2}dx=-\int_0^1\int_0^1 \frac{x^2\ln(x)\ln(y)}{(1+x^2)(1-xy)}dydx$$

$$=-\left(\int_0^1\int_0^1 \frac{\ln(x)\ln(y)}{(1-xy)}dydx-\int_0^1\int_0^1 \frac{\ln(x)\ln(y)}{(1+x^2)(1+y^2)}dydx-\int_0^1\int_0^1 \frac{xy\ln(x)\ln(y)}{(1+x^2)(1+y^2)}dydx\right)$$

$$=\frac{-1}{2}\left(\int_0^1\int_0^1 \frac{\ln(x)\ln(y)}{(1-xy)}dydx-\left(\int_0^1 \frac{\ln(x)}{(1+x^2)}dx\right)^2-\left(\int_0^1 \frac{x\ln(x)}{(1+x^2)}dx\right)^2\right)$$

$$=\frac{-1}{2}\left(\sum_{n=1}^{\infty}\int_0^1\int_0^1 x^{n-1}y^{n-1}\ln(x)\ln(y)dydx-\left(\sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1 \mathrm{x}^{2\mathrm{n}-2}\ln(x)\,dx\right)^2\right.$$
$$\left.-\left(\sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1 \mathrm{x}^{2\mathrm{n}-1}\ln(x)\,dx\right)^2\right)$$

$$=\frac{-1}{2}\left(\sum_{n=1}^{\infty}\frac{1}{n^4}-\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\right)^2-\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)^2}\right)^2\right)$$

$$=\frac{-1}{2}\left(\sum_{n=1}^{\infty}\frac{1}{n^4}-\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\right)^2-\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)^2}\right)^2\right)=\frac{-1}{2}\left(\zeta(4)-G^2-\left(\frac{1}{8}\zeta(2)\right)^2\right)=$$

$$=\frac{1}{2}\left(G^2-\frac{123}{128}\zeta(4)\right)$$

$$B=\int_0^1\frac{\ln(x)\,Li_2(x)}{1+x}dx=-\int_0^1\int_0^1\frac{x\ln(x)\ln(y)}{(1+x)(1-xy)}dydx=$$

$$=-\frac{1}{2}\left(\int_0^1\int_0^1\frac{x\ln(x)\ln(y)}{(1+x)(1-xy)}dydx+\int_0^1\int_0^1\frac{y\ln(x)\ln(y)}{(1+y)(1-xy)}dydx\right)$$

$$=-\frac{1}{2}\int_0^1\int_0^1\frac{(x+y+2xy)\ln(x)\ln(y)}{(1+y)(1+x)(1-xy)}dydx$$

$$=-\frac{1}{2}\int_0^1\int_0^1\frac{[(1+x)(1+y)-(1-xy)]\ln(x)\ln(y)}{(1+y)(1+x)(1-xy)}dydx$$

$$=-\frac{1}{2}\left(\int_0^1\int_0^1\frac{\ln(x)\ln(y)}{(1-xy)}dydx-\int_0^1\int_0^1\frac{\ln(x)\ln(y)}{(1+y)(1+x)}dydx\right)$$

$$=-\frac{1}{2}\left(\int_0^1\int_0^1\frac{\ln(x)\ln(y)}{(1-xy)}dydx-\left(\int_0^1\frac{\ln(x)}{1+x}dx\right)^2\right)$$

$$=\frac{-1}{2}\left(\sum_{n=1}^{\infty}\int_0^1\int_0^1 x^{n-1}y^{n-1}\ln(x)\ln(y)dydx-\left(\sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1 \mathrm{x}^{n-1}\ln(x)\,dx\right)^2\right)$$

$$=\frac{-1}{2}\left(\sum_{n=1}^{\infty}\frac{1}{n^4}-\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\right)^2\right)=\frac{-1}{2}\left(\zeta(4)-\left(\frac{1}{2}\zeta(2)\right)^2\right)=\frac{-1}{2}\left(\frac{\pi^4}{90}-\frac{\pi^4}{144}\right)$$

$$=\frac{-1}{2}\left(\frac{\pi^4}{90}\left(\frac{54}{144}\right)\right)=-\frac{3}{16}\zeta(4)$$

$$I = 2A - \frac{1}{8}B, \quad I = 2\left(\frac{1}{2}\left(G^2 - \frac{123}{128}\zeta(4)\right)\right) - \frac{1}{8}\left(-\frac{3}{16}\zeta(4)\right)$$

$$I = G^2 - \frac{120}{128}\zeta(4), \quad I = G^2 - \frac{120}{128}\frac{\pi^4}{90}, \quad I = G^2 - \frac{\pi^4}{96}$$

Or,

$$\int_0^1 \frac{x\ln(x)\{Li_2(x) - Li_2(-x)\}}{1+x^2}dx = G^2 - \frac{\pi^4}{96}$$