ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega=\int_0^1 \frac{1}{\sqrt{x}} ln\left(\sum_{m=0}^{n} x^{2m}\right)dx$$

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Solution by Pratham Prasad-India

$$\Omega=\int_0^1 2ln\left(\sum_{m=0}^{n} x^{4m}\right)dx=2\left(\int_0^1 ln\big(1-x^{4(n+1)}\big)dx-\int_0^1 ln(1-x^4)dx\right)=$$

$$=2(I(4(n+1))-I(4))$$

Thus

$$I=2(I(4(n+1))-I(4))$$

where

$$I(a)=\int_0^1 ln(1-x^a)dx,\ \ x^a=t,\qquad dx=\frac{1}{a}t^{\frac{1}{a}-1}dt$$

$$aI(a)=\int_0^1 t^{\frac{1}{a}-1}ln(1-t)dx,\ \ aI(a)=\lim_{u\to 1}\frac{d}{du}B(\frac{1}{a},u)$$

$$aI(a)=\lim_{u\to 1}B(\frac{1}{a},u)\left(\psi(u)-\psi(u+\frac{1}{a})\right),\ \ aI(a)=a\left(\psi(1)-\psi\left(\frac{1}{a}\right)-a\right)$$

$$I(a)=\left(\psi(1)-\psi\left(\frac{1}{a}\right)-a\right)$$

$$I=2\left(\psi(1)-\psi\left(\frac{1}{4(n+1)}\right)-4(n+1)-\psi(1)+\psi\left(\frac{1}{4}\right)+4\right)$$

$$I=2\left(\psi\left(\frac{1}{4}\right)-\psi\left(\frac{1}{4(n+1)}\right)-4n\right)$$