ROMANIAN MATHEMATICAL MAGAZINE

Prove that

$$I=\int_0^1\int_0^1\left(\frac{x-y}{x+y}\right)\frac{1}{\log\left(\frac{x}{y}\right)}dx\,dy=\log\left(\frac{\pi}{2}\right)$$

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$$I=\int_0^1\int_0^1\left(\frac{\frac{x}{y}-1}{\frac{x}{y}+1}\right)\frac{1}{\log\left(\frac{x}{y}\right)}dx\,dy$$

$$=\int_0^1 y\int_0^{\frac{1}{y}}\left(\frac{z-1}{z+1}\right)\frac{1}{\log(z)}dz\,dy\overset{IBP}{=}\left[\frac{y^2}{2}\int_0^{\frac{1}{y}}\left(\frac{z-1}{z+1}\right)\frac{1}{\log(z)}dz\right]_{y=0}^{y=1}$$

$$-\frac{1}{2}\int_0^1\left(\frac{1-y}{1+y}\right)\frac{1}{\log(y)}dy=\frac{1}{2}\int_0^1\left(\frac{z-1}{z+1}\right)\frac{1}{\log(z)}dz-\frac{1}{2}\int_0^1\left(\frac{1-y}{1+y}\right)\frac{1}{\log(y)}dy$$

$$=\int_0^1\left(\frac{z-1}{z+1}\right)\frac{1}{\log(z)}dz=\int_0^1\int_0^1\frac{z^m}{1+z}dm\,dz=\sum_{k=0}^{\infty}(-1)^k\int_0^1\int_0^1 z^{m+k}\,dz\,dm$$

$$=\sum_{k=0}^{\infty}(-1)^k\left(\log(k+2)-\log(k+1)\right)=\sum_{k=1}^{\infty}(-1)^{k-1}\log\left(\frac{k+1}{k}\right)$$

$$=2\eta'(0)=\log\left(\frac{\pi}{2}\right)$$

Note: $\eta'(0)=-\frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\log\left(\frac{k}{k+1}\right)$