ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{dxdydz}{(1+x^2y^2)(1+x^2z^2)(1+y^2z^2)}=\frac{\pi^3}{4\sqrt{2}}$$

Proposed by Ankush Kumar Parcha-India

Solution 1 by Cosghun Memmedov-Azerbaijan

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{dxdydz}{(1+x^2y^2)(1+x^2z^2)(1+y^2z^2)}\overset{x\to 1/x}{\cong}\int_0^\infty\int_0^\infty\int_0^\infty \frac{dxdydz}{(x^2+y^2)(x^2+z^2)(y^2+z^2)}=$$

$$\int_0^\infty\int_0^\infty\int_0^\infty \left(\frac{z^2}{x^2+z^2}-\frac{y^2}{x^2+y^2}\right)\frac{dxdydz}{(1+z^2y^2)(z^2-y^2)}$$

$$=\int_0^\infty\int_0^\infty \frac{1}{(1+z^2y^2)(z^2-y^2)}\left(z\tan^{-1}\left(\frac{x}{z}\right)-y\tan^{-1}\left(\frac{x}{y}\right)\right)\Big|_0^\infty dydz$$

$$\frac{\pi}{2}\int_0^\infty\int_0^\infty \frac{1}{(y+z)(1+z^2y^2)}dydz\overset{IBP}{\cong}$$

$$=\frac{\pi}{2}\int_0^\infty\int_0^\infty \frac{\tan^{-1}(yz)}{z(y+z)^2}dydz\overset{(By\ symmetry)}{\cong}\frac{\pi}{4}\int_0^\infty\int_0^\infty \frac{\tan^{-1}(yz)}{zy(y+z)}dydz$$

$$\underset{\left(\frac{dt}{dy}=z\right)}{\overset{(yz=t)}{\cong}}\frac{\pi}{4}\int_0^\infty\int_0^\infty \frac{\tan^{-1}(t)}{t(t+z^2)}dydz=\frac{\pi}{4}\int_0^\infty \frac{\tan^{-1}}{t}dt\int_0^\infty \frac{1}{z^2+t}dz=\frac{\pi^2}{8}\int_0^\infty \frac{\tan^{-1}(t)}{t^{\frac{3}{2}}}dt=$$

$$-\frac{\pi^2}{4}\int_0^\infty \tan^{-1}(t)d\left(t^{-\frac{1}{2}}\right)\overset{IBP}{\cong}\frac{\pi^2}{4}\int_0^\infty \frac{t^{-\frac{1}{2}}}{1+t^2}dt=\frac{\pi^2}{2}\int_0^\infty \frac{1}{1+t^4}dt=\frac{\pi^3}{4\sqrt{2}}$$

Solution 2 by Pham Duc Nam-Vietnam

$$I=\int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{(1+x^2y^2)(1+y^2z^2)(1+x^2z^2)}dxdydz=\frac{\pi^3}{4\sqrt{2}}?$$

$$=\frac{1}{2}\int_0^\infty\int_0^\infty \frac{1}{1+y^2z^2}dydz\int_{-\infty}^\infty \frac{1}{(1+x^2y^2)(1+x^2z^2)}dx$$

$$=\int_0^\infty\int_0^\infty \frac{1}{1+y^2z^2}dydz\left(\pi i.\,Res\left(\frac{1}{(1+x^2y^2)(1+x^2z^2)},x=\frac{i}{y},x=\frac{i}{z}\right)\right)$$

$$=\int_0^\infty\int_0^\infty \frac{1}{1+y^2z^2}dydz\left(\pi i.\left(\begin{array}{l}\lim\limits_{x\to\frac{i}{y}}\left(x-\frac{i}{y}\right)\frac{1}{(1+x^2y^2)(1+x^2z^2)}+\\ +\lim\limits_{x\to\frac{i}{z}}\left(x-\frac{i}{z}\right)\frac{1}{(1+x^2y^2)(1+x^2z^2)}\end{array}\right)\right)$$

$$=\int_0^\infty\int_0^\infty\frac{1}{1+y^2z^2}dydz\left(\pi i\left(-\frac{i}{2(y+z)}\right)\right)=\frac{\pi}{2}\int_0^\infty\int_0^\infty\frac{1}{(y+z)(1+y^2z^2)}dydz$$

$$t=yz\Rightarrow y=\frac{t}{z}\Rightarrow dy=\frac{1}{z}dt\Rightarrow I=\frac{\pi}{2}\int_0^\infty\int_0^\infty\frac{1}{\left(\frac{t}{z}+z\right)(1+t^2)}\frac{1}{z}dt\,dz=$$

$$=\frac{\pi}{z}\int_0^\infty\int_0^\infty\frac{1}{(t+z^2)(1+t^2)}dtdz$$

$$=\frac{\pi}{2}\int_0^\infty\frac{1}{1+t^2}dt\int_0^\infty\frac{1}{t+z^2}dz=\frac{\pi}{2}\int_0^\infty\frac{1}{1+t^2}dt\left(\frac{\pi}{2}\frac{1}{\sqrt{t}}\right)=\frac{\pi^2}{4}\int_0^\infty\frac{1}{\sqrt{t}(1+t^2)}dt=$$

$$=\frac{\pi^2}{2}\int_0^\infty\frac{1}{2\sqrt{t}(1+t^2)}dt$$

$$=\frac{\pi^2}{2}\int_0^\infty\frac{1}{(1+t^2)}d\left(\sqrt{t}\right)=\frac{\pi^2}{2}\int_0^\infty\frac{1}{1+t^4}dt=\frac{\pi^2}{4}\int_{-\infty}^\infty\frac{t^2}{1+t^4}dt=$$

$$=\frac{\pi^2}{4}\int_{-\infty}^\infty\frac{1}{\left(t-\frac{1}{t}\right)^2+2}dt\overset{Glasser's\ master\ theorem}{\rightarrow}\frac{\pi^2}{4}\int_{-\infty}^\infty\frac{1}{t^2+2}dt$$

$$=\frac{\pi^2}{4}\frac{1}{\sqrt{2}}\arctan\left(\frac{1}{\sqrt{2}}t\right)\Big|_{-\infty}^{\infty}=\frac{\pi^3}{4\sqrt{2}}$$