ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Omega=\int_0^1\int_0^1 ln\left(\frac{x\sqrt{x}}{y\sqrt{y}}+\frac{y\sqrt{y}}{x\sqrt{x}}\right)dxdy$$

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$$\Omega=\int_0^1\int_0^1 \ln\left(\frac{x\sqrt{x}}{y\sqrt{y}}+\frac{y\sqrt{y}}{x\sqrt{x}}\right)dxdy=$$

$$=\int_0^1\int_0^1 \ln(x^3+y^3)dxdy-\frac{3}{2}\int_0^1\int_0^1 \ln(xy)dxdy=M-\frac{3}{2}K$$

$$M=\int_0^1\int_0^1 ln(x^3+y^3)dxdy=\int_0^1 \ln(1+y^3)dy-3\int_0^1\int_0^1\frac{x^3}{x^3+y^3}dxdy=P-3Q$$

$$P=\int_0^1 \ln(1+y^3)dy=\ln 2-3\int_0^1\frac{y^3}{1+y^3}dy=\ln(2)-3+3\int_0^1\frac{1}{1+y^3}dy=$$

$$=\ln(2)-3+3(\frac{1}{3}\ln(2)+\frac{\sqrt{3}}{9}\pi=2\ln(2)+\frac{\pi}{\sqrt{3}}-3$$

$$Q=\int_0^1\int_0^1\frac{x^3}{x^3+y^3}dxdy=\int_0^1\int_0^1\frac{y^3}{x^3+y^3}dx\,dy$$

$$2Q=\int_0^1\int_0^1\frac{x^3+y^3}{x^3+Y^3}dx\,dy=1\ \leftrightarrow\ Q=\frac{1}{2}$$

$$M=P-3Q=2\ln(2)+\frac{\pi}{\sqrt{3}}-3-\frac{3}{2}=2\ln(2)+\frac{\pi}{\sqrt{3}}-\frac{9}{2}$$

$$K=\int_0^1\int_0^1\ln(xy)\,dxdy=\int_0^1\int_0^1\ln(x)\,dxdy+\int_0^1\int_0^1\ln(y)\,dxdy=-2$$

$$\Omega=M-\frac{3}{2}K=2\ln(2)+\frac{\pi}{\sqrt{3}}-\frac{9}{2}+\frac{3}{2}x2=2\ln(2)+\frac{\pi}{\sqrt{3}}-\frac{3}{2}$$