ROMANIAN MATHEMATICAL MAGAZINE

Prove the below closed form

$$\int_0^1 \frac{\tan^{-1}(x)}{\sqrt{1+x}}dx = \frac{\pi}{\sqrt{2}} - \sqrt{2+2\sqrt{2}}\tan^{-1}\left(\frac{2\sqrt{2+10\sqrt{2}}}{7}\right)$$

Proposed by Ankush Kumar Parcha-India

Solution 1 by Rana Ranino-Algerie

$$\Omega = \int_0^1 \frac{\tan^{-1} x}{\sqrt{1+x}}dx$$

$$\Omega = \left[2\tan^{-1}(x)\sqrt{1+x}\right]_0^1 - 2\int_0^1 \frac{\sqrt{1+x}}{1+x^2}dx = \frac{\pi}{\sqrt{2}} - 2\int_0^1 \frac{\sqrt{1+x}}{1+x^2}dx$$

$$\Omega \overset{1+x=t^2}{=} \frac{\pi}{\sqrt{2}} - 4\int_1^{\sqrt{2}} \frac{t^2}{t^4-2t^2+2}dt =$$

$$= \frac{\pi}{\sqrt{2}} - 2\int_1^{\sqrt{2}} \frac{t^2+\sqrt{2}}{t^4-2t^2+2}dt - 2\int_0^{\sqrt{2}} \frac{t^2-\sqrt{2}}{t^4-2t^2+2}dt$$

$$\Omega = \frac{\pi}{\sqrt{2}} - 2\int_1^{\sqrt{2}} \frac{\left(1+\frac{\sqrt{2}}{t^2}\right)}{\left(t-\frac{\sqrt{2}}{t}\right)^2 - 2 + 2\sqrt{2}}dt - 2\int_1^{\sqrt{2}} \frac{\left(1-\frac{\sqrt{2}}{t^2}\right)}{\left(t+\frac{\sqrt{2}}{t}\right)^2 - 2 - 2\sqrt{2}}dt$$

$$\Omega = \frac{\pi}{\sqrt{2}} - 2\int_{1-\sqrt{2}}^{\sqrt{2}-1} \frac{1}{u^2+2(\sqrt{2}-1)}du - 2\int_{1+\sqrt{2}}^{1-\sqrt{2}} \frac{1}{v^2-2(2\sqrt{2}+1)}dv =$$

$$= \frac{\pi}{\sqrt{2}} - 4\int_0^{\sqrt{2}-1} \frac{1}{u^2+2(\sqrt{2}-1)}du$$

$$\Omega = \frac{\pi}{\sqrt{2}} - 4\left[\frac{1}{\sqrt{2(\sqrt{2}-1)}}\tan^{-1}\left(\frac{u}{\sqrt{2(\sqrt{2}-1)}}\right)\right]_0^{\sqrt{2}-1} =$$

$$= \frac{\pi}{\sqrt{2}} - 2\sqrt{2\sqrt{2}+2}\tan^{-1}\left(\sqrt{\frac{\sqrt{2}-1}{2}}\right)$$

Using identity: $2\tan^{-1}(a) = \tan^{-1}\left(\frac{2a}{1-a^2}\right)$

$$\Omega=\frac{\pi}{\sqrt{2}}-\sqrt{2\sqrt{2}+2}\tan^{-1}\left(\frac{2\sqrt{\frac{\sqrt{2}-1}{2}}}{1-\frac{\sqrt{2}-1}{2}}\right)=\frac{\pi}{\sqrt{2}}-\sqrt{2\sqrt{2}+2}\tan^{-1}\left(\frac{4\sqrt{\sqrt{2}-1}}{3\sqrt{2}-2}\right)$$

$$\Omega=\frac{\pi}{\sqrt{2}}-\sqrt{2\sqrt{2}+2}\tan^{-1}\left(\frac{2\sqrt{(\sqrt{2}-1)(22+12\sqrt{2})}}{7}\right)=$$

$$=\frac{\pi}{\sqrt{2}}-\sqrt{2\sqrt{2}+2}\tan^{-1}\left(\frac{2\sqrt{2+10\sqrt{2}}}{7}\right)$$

$$\int_0^1\frac{\tan^{-1}x}{\sqrt{1+x}}dx=\frac{\pi}{\sqrt{2}}-\sqrt{2\sqrt{2}+2}\tan^{-1}\left(\frac{2\sqrt{2+10\sqrt{2}}}{7}\right)$$

Solution 2 by Cosghun Memmedov-Azerbaijan

$$\Omega=\int_0^1\frac{\tan^{-1}x}{\sqrt{x+1}}dx\overset{IBP}{=}2\left(\tan^{-1}(x)\sqrt{x+1}\right)\Big|_0^1-2\int_0^1\frac{\sqrt{x+1}}{x^2+1}dx=\frac{\pi}{\sqrt{2}}-2A$$

$$A=\int_0^1\frac{\sqrt{x+1}}{x^2+1}dx\overset{\sqrt{x+1}=t}{=}2\int_0^{\sqrt{2}}\frac{t^2}{t^4-2t^2+2}dt\overset{t\to\frac{\sqrt{2}}{t}}{=}2\int_0^{\sqrt{2}}\frac{\sqrt{2}}{t^4-2t^2+2}dt=$$

$$=\int_0^{\sqrt{2}}\frac{t^2}{t^4-2t^2+2}dt=\int_0^{\sqrt{2}}\frac{\sqrt{2}}{t^4-2t^2+2}dt=|\to 2|=\int_0^{\sqrt{2}}\frac{t^2+\sqrt{2}}{t^4-2t^2+2}dt$$

$$A=2|=\int_0^{\sqrt{2}}\frac{t^2+\sqrt{2}}{t^4-2t^2+2}dt=\int_0^{\sqrt{2}}\frac{\left(1+\frac{\sqrt{2}}{t^2}\right)}{t^2+\left(\frac{\sqrt{2}}{t}\right)^2-2}dt=$$

$$=\int_0^{\sqrt{2}}\frac{d\left(t-\frac{\sqrt{2}}{t}\right)}{\left(t-\frac{\sqrt{2}}{t}\right)^2+\left(\sqrt{2\sqrt{2}-2}\right)^2}=\frac{1}{\sqrt{2\sqrt{2}-2}}\tan^{-1}\left(\frac{\left(t-\frac{\sqrt{2}}{t}\right)}{\sqrt{2\sqrt{2}-2}}\right)\Bigg|_1^{\sqrt{2}}=$$

$$=\frac{\sqrt{2+\sqrt{2}}}{2}\left(\tan^{-1}\frac{\sqrt{2}-1}{\sqrt{2\sqrt{2}-2}}\right)-\tan^{-1}\left(\frac{1-\sqrt{2}}{\sqrt{2\sqrt{2}-2}}\right)=$$

$$=\frac{\sqrt{2+\sqrt{2}}}{2}\tan^{-1}\left(\frac{\frac{2\sqrt{2}-1}{\sqrt{2\sqrt{2}-2}}}{1-\frac{\left(\sqrt{2}-1\right)^2}{2\sqrt{2}-2}}\right)=\frac{\sqrt{2+\sqrt{2}}}{2}\tan^{-1}\left(\frac{\sqrt{(2\sqrt{2}-2)(12-8\sqrt{2})}}{4\sqrt{2}-5}\right)=$$

$$\Omega=\frac{\pi}{\sqrt{2}}-2A=\frac{\pi}{\sqrt{2}}-\sqrt{2+2\sqrt{2}}\tan^{-1}\left(\frac{2\sqrt{2+10\sqrt{2}}}{7}\right)$$

Notes: $\tan^{-1}(x)\pm\tan^{-1}(y)=\tan\left(\frac{x\pm y}{1\mp xy}\right)$

Answer:

$$\int_0^1\frac{\tan^{-1}x}{\sqrt{x+1}}dx=\frac{\pi}{\sqrt{2}}-\sqrt{2+2\sqrt{2}}\tan^{-1}\left(\frac{2\sqrt{2+10\sqrt{2}}}{7}\right)$$