ROMANIAN MATHEMATICAL MAGAZINE

Prove that :

$$\int_0^1 \frac{x}{\sinh(x)}dx = Li_2\left(-\frac{1}{e}\right) - Li_2\left(\frac{1}{e}\right) - \ln(\frac{e+1}{e-1}) + \frac{\pi^2}{4}$$

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$$\omega = \int_0^1 \frac{x}{\sinh(x}dx = 2\int_0^1 \frac{x\,e^x}{e^{2x}-1}dx \quad \left\{e^{-x} = t;\ x = -\ln(t)\,; \frac{dx}{dt} = -\frac{1}{t}\right\}$$

$$\omega = 2\int_1^{\frac{1}{e}} \frac{\frac{1}{t}\ln(t)}{(\frac{1}{t^2}-1)t}dt = 2\int_1^{\frac{1}{e}} \frac{\ln(t)}{1-t^2}dt =$$

$$= 2\left(\int_0^{\frac{1}{e}} \frac{\ln(t)}{1-t^2}dt - \int_0^1 \frac{\ln(t)}{1-t^2}dt\right) = 2(\omega_1 - \omega_2)$$

ω_1

$$= \int_0^{\frac{1}{e}} \frac{\ln(t)}{1-t^2}dt = \sum_{n=0}^{\infty}\int_0^{\frac{1}{e}} t^{2n}\ln(t)dt \underset{\text{İBP}}{=} \sum_{n=0}^{\infty}\left(-\frac{1}{(2n+1)e^{2n}+1} - \frac{1}{(2n+1)^2 e^{2n+1}}\right) =$$

$$= -\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{1}{ne^n} - \sum_{n=1}^{\infty}\frac{(-1)^n}{ne^n} + \sum_{n=1}^{\infty}\frac{1}{n^2e^n} - \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2e^n}\right) =$$

$$= -\frac{1}{2}\left(1 - \ln(e-1) - 1 + \ln(e+1) + Li_2\left(\frac{1}{e}\right) - Li_2\left(-\frac{1}{e}\right)\right)$$

$$= -\frac{1}{2}\ln\left(\frac{e+1}{e-1}\right) - \frac{1}{2}Li_2\left(\frac{1}{e}\right) + \frac{1}{2}Li_2\left(-\frac{1}{e}\right)$$

$$\omega_2 = \int_0^1 \frac{\ln(t)}{1-t^2}dt =$$

$$= \sum_{n=0}^{\infty}\int_0^1 t^{2n}\ln(t) \underset{\text{İBP}}{=} -\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = -\frac{1}{2}(\sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}) =$$

$$= -\frac{1}{2}\left(\zeta(2) - \frac{1}{2}\zeta(2)\right) = -\frac{1}{2}\left(\frac{\pi^2}{6} + \frac{\pi^2}{12}\right) = -\frac{\pi^2}{8}\ ;$$

$$\omega = 2(\omega_1 - \omega_2) = 2\left(-\frac{1}{2}\ln\left(\frac{e+1}{e-1}\right) - \frac{1}{2}Li_2\left(\frac{1}{e}\right) + \frac{1}{2}Li_2\left(-\frac{1}{e}\right) + \frac{\pi^2}{8}\right)$$

$$= Li_2\left(-\frac{1}{e}\right) - Li_2\left(\frac{1}{e}\right) - \ln\left(\frac{e+1}{e-1}\right) + \frac{\pi^2}{4}$$

$$\int_0^1 \frac{x}{sinh(x)}dx = Li_2\left(-\frac{1}{e}\right) - Li_2\left(\frac{1}{e}\right) - \ln(\frac{e+1}{e-1}) + \frac{\pi^2}{4}$$

Notes:

$$\sum_{n=0}^{\infty} a_{2n+1} = \frac{1}{2}\left(\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty}(-1)^n a_n\right);\ sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$Li_s(n) = \sum_{n=1}^{\infty} \frac{x^n}{n^s}\ Polylogarithm\ function$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}\quad Rieman's\ zeta\ function$$