ROMANIAN MATHEMATICAL MAGAZINE

Prove that

$$\int_{R^+}\left(\frac{\sin(x)}{\sinh(x)}\right)\left(\frac{\cos(x)}{\cosh(x)}\right)dx=\frac{\pi}{4}\tanh\left(\frac{\pi}{2}\right)$$

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$$\sigma=\int_0^\infty\frac{\sin(x)\cos(x)}{\sinh(x)\cosh(x)}dx=\int_0^\infty\frac{\sin(2x)}{\sinh(2x)}dx=\frac{1}{2}\int_0^\infty\frac{\sin(x)}{\sinh(x)}dx=\int_0^\infty\frac{e^{-x}\sin(x)}{1-e^{-2x}}dx$$

$$=\int_0^\infty\sum_{n=0}^\infty e^{-2nx-x}\sin(x)\,dx=\sum_{n=0}^\infty\int_0^\infty e^{-x(2n+1)}\sin(x)dx=\sum_{n=0}^\infty(f(2n+1))$$

$$f(a)=\int_0^\infty e^{-ax}\sin(x)dx,\quad \textit{Using IBP method}\ \left\{u=\sin(x),\ du=\cos(x)\,dx;\ v=\int e^{-ax}dx=-\frac{1}{a}e^{-ax}\right\}$$

$$f(a)=\left[-\frac{e^{-ax}\sin(x)}{a}\right]_0^\infty+\frac{1}{a}\int_0^\infty e^{-ax}\cos(x)dx=\frac{1}{a}\left[-\frac{e^{-ax}\cos(x)}{a}\right]_0^\infty-\frac{1}{a^2}\int_0^\infty e^{-ax}\sin(x)\,dx$$

$$f(a)=\frac{1}{a^2}-\frac{1}{a^2}f(a)\,;\ f(a)=\frac{1}{a^2+1},f(2n+1)=\frac{1}{1+(2n+1)^2}$$

$$\sigma=\sum_{n=0}^\infty f(2n+1)=\sum_{n=0}^\infty\frac{1}{(2n+1)^2+1}=\sum_{n=0}^\infty\frac{1}{(2n+1-i)(2n+1+i)}$$

$$=\frac{1}{4}\sum_{n=0}^\infty\frac{1}{\left(n+\frac{1-i}{2}\right)\left(n+\frac{1+i}{2}\right)}=$$

$$=\frac{1}{4}\left(\frac{\psi^{(0)}\left(\frac{1+i}{2}\right)-\psi^{(0)}\left(\frac{1-i}{2}\right)}{i}\right)=\frac{1}{4i}\left(\psi^{(0)}\left(\frac{1+i}{2}\right)-\psi^{(0)}\left(1-\frac{1+i}{2}\right)\right)=$$

$$=-\frac{1}{4i}\left(\pi\cot\left(\pi\frac{1+i}{2}\right)\right)=\frac{\pi}{4i}\tan\left(\frac{\pi i}{2}\right)=\frac{\pi}{4}\tanh\left(\frac{\pi}{2}\right)$$

***Notes*:**

***Polygamma reflection formula*:**

$$(-1)^m\psi^{(m)}(1-z)-\psi^{(m)}(z)=\pi\frac{d^m}{dz^m}\cot z)$$

***Hiperbolic tangent function*:**

$$\tanh(x)=\frac{e^x-e^{-x}}{e^x+e^{-x}};\ \tan(x)=\frac{i(e^{-ix}-e^{ix})}{e^{-ix}+e^{ix}};\ \tanh(ix)=i\tan(x)$$