ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^1 \frac{\sinh^{-1}(x)}{x}dx = -\frac{Li_2(3-2\sqrt{2})}{2} + \frac{\pi^2}{12} - \frac{ln^2(1+\sqrt{2})}{2} + \ln(1+\sqrt{2})\ln(2)$$

Proposed by Ankush Kumar Parcha-India

Solution by Amin Hajiyev-Azerbaijan

$$\Omega = \int_0^1 \frac{\sinh^{-1}(x)}{x}dx = \int_0^1 \frac{\ln\left(\sqrt{1+x^2}+x\right)}{x}dx = -\int_0^1 \frac{\ln\left(\sqrt{1+x^2}-x\right)}{x}dx$$

$$substitution\ \left\{\sqrt{1+x^2}-x = t, x = \frac{1-t^2}{2t}, dx = -\frac{1+t^2}{2t^2}dt, t[\sqrt{2}-1;1]\right\}$$

$$\Omega = -\int_{\sqrt{2}-1}^1 \frac{\ln(t)}{\left(\frac{1-t^2}{2t}\right)}\frac{(1+t^2)dt}{2t^2} = -\int_{\sqrt{2}-1}^1 \frac{(1+t^2)\ln(t)}{t(1-t^2)}dt =$$

$$= -\int_{\sqrt{2}-1}^1 \frac{\ln(t)}{t(1-t^2)}dt - \int_{\sqrt{2}-1}^1 \frac{tln(t)}{1-t^2}dt = -(\Omega_1+\Omega_2)$$

$$\Omega_1 = \int_{\sqrt{2}-1}^1 \frac{\ln(t)}{t(1-t^2)}dt = \int_{\sqrt{2}-1}^1 \frac{\ln(t)}{t}dt + \int_{\sqrt{2}-1}^1 \frac{tln(t)}{1-t^2}dt = \int_{\sqrt{2}-1}^1 \frac{\ln(t)}{t}dt + \Omega_2$$

$$\Omega_1 = [\frac{ln^2(t)}{2}]_{\sqrt{2}-1}^1 + \sum_{n=0}^{\infty}\int_{\sqrt{2}-1}^1 t^{2n+1}\ln(t)\,dt =$$

$$= -\frac{ln^2(1+\sqrt{2})}{2} + \sum_{n=0}^{\infty}\left(\frac{t^{2n+2}}{2n+2}\ln(t)|_{\sqrt{2}-1}^1 - \frac{1}{2n+2}\int_{\sqrt{2}-1}^1 t^{2n+1}\,dt\right) =$$

$$= -\frac{ln^2(1+\sqrt{2})}{2} - \frac{\ln(\sqrt{2}-1)}{2}\sum_{n=1}^{\infty}\frac{(\sqrt{2}-1)^{2n}}{n} - \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{(\sqrt{2}-1)^{2n}}{n^2} =$$

$$= -\frac{ln^2(1+\sqrt{2})}{2} + \frac{\ln(\sqrt{2}-1)\ln(2\sqrt{2}-2)}{2} - \frac{\pi^2}{24} + \frac{1}{4}\mathrm{Li}_2(3-2\sqrt{2}) =$$

$$= -\frac{ln^2(1+\sqrt{2})}{2} + \frac{ln^2(1+\sqrt{2})}{2} - \frac{\ln(1+\sqrt{2})\ln(2)}{2} - \frac{\pi^2}{24} + \frac{1}{4}Li_2(3-2\sqrt{2}) =$$

$$= \frac{1}{4}Li_2(3-2\sqrt{2}) - \frac{\pi^2}{24} - \frac{\ln(1+\sqrt{2})\ln(2)}{2}$$

$$\Omega_2 = \Omega_1 + \frac{ln^2(1+\sqrt{2})}{2} = \frac{1}{4}Li_2(3-2\sqrt{2}) - \frac{\pi^2}{24} - \frac{\ln(1+\sqrt{2})\ln(2)}{2} + \frac{ln^2(1+\sqrt{2})}{2}$$

$$\Omega = -(\Omega_1+\Omega_2) = -\frac{1}{2}Li_2(3-2\sqrt{2}) + \frac{\pi^2}{12} + \ln(1+\sqrt{2})\ln(2) - \frac{ln^2(1+\sqrt{2})}{2}$$

$$\int_0^1 \frac{\sinh^{-1}(x)}{x}dx = -\frac{Li_2(3-2\sqrt{2})}{2} + \frac{\pi^2}{12} - \frac{ln^2(1+\sqrt{2})}{2} + \ln(1+\sqrt{2})\ln(2)$$