ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^1 \sinh^{-1}(x)\cosh^{-1}(x)dx = \frac{\pi}{2}\left(1 - \frac{4\sqrt{2\pi}}{\Gamma^2(1/4)}\right)i$$

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$$\int_0^1 \sinh^{-1}(x)\cosh^{-1}(x)dx = \int_0^1 ln(x+\sqrt{x^2+1})ln(x+\sqrt{x^2-1})dx \overset{IBP}{\triangleq}$$

$$=-\left(\int_0^1 \frac{xln(x+\sqrt{x^2-1})}{\sqrt{x^2+1}}dx + \int_0^1 \frac{xln(x+\sqrt{x^2+1})}{\sqrt{x^2-1}}dx\right) \overset{IBP}{\triangleq} \ln(i) + \int_0^1 \frac{\sqrt{x^2+1}}{\sqrt{x^2-1}}dx + \int_0^1 \frac{\sqrt{x^2-1}}{\sqrt{x^2-1}}dx =$$

$$=\frac{\pi i}{2} - 2i\int_0^1 \frac{x}{\sqrt{1-x^4}}dx \overset{(x^4\to x)}{\triangleq} \frac{\pi i}{2} - \int_0^1 x^{-\frac{1}{4}}(1-x)^{-\frac{1}{2}}dx = \frac{\pi i}{2} - \frac{i}{2}B\left(\frac{3}{4},\frac{1}{2}\right) =$$

$$=\frac{\pi i}{2} - \frac{i}{2}\times\frac{\Gamma(\frac{3}{4})\times\Gamma(\frac{1}{2})}{\Gamma(\frac{5}{4})} = \frac{\pi i}{2} - \frac{\pi i}{2}\times\frac{4\sqrt{2\pi}}{\Gamma^2(\frac{1}{4})} = \frac{\pi}{2}\left(1 - \frac{4\sqrt{2\pi}}{\Gamma^2(1/4)}\right)i$$

Notes: $\Gamma(\frac{5}{4}) = \Gamma(1+\frac{1}{4}) = \frac{1}{4}\Gamma(\frac{1}{4})$; $\Gamma(\frac{3}{4}) = \frac{\pi\sqrt{2}}{\Gamma(\frac{1}{4})}$; $\Gamma(\frac{1}{2}) = \sqrt{\pi}$