ROMANIAN MATHEMATICAL MAGAZINE

If $I=\int_0^1\int_0^1\int_0^1\sum_{x,y,z}\log(x)\tan^{-1}(xyz)\,dx\,dy\,dz$ then, show that:

$$I=\frac{9}{32}\zeta(3)+\frac{7\pi^4}{3840}+\frac{\pi^2}{16}-\frac{3\pi}{4}+\frac{3}{2}\log(2)$$

where, $\zeta(3)$ is an Apery's constant.

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Solution 1 by Togrul Ehmedov-Azerbaijan

$$I=\int_0^1\int_0^1\int_0^1\sum_{x,y,z}\log(x)\tan^{-1}(xyz)\,dx\,dy\,dz=\int_0^1\int_0^1\int_0^1\log(xyz)\tan^{-1}(xyz)\,dx\,dy\,dz$$

$$=\int_0^1\frac{1}{x}\int_0^1\frac{1}{y}\int_0^{xy}\log(m)\tan^{-1}(m)\,dm\,dy\,dx\overset{IBP}{=}$$

$$-\int_0^1\log(y)\int_0^1\log(xy)\tan^{-1}(xy)\,dx\,dy$$

$$=-\int_0^1\frac{\log(y)}{y}\int_0^y\log(p)\tan^{-1}(p)\,dp\,dy\overset{IBP}{=}\frac{1}{2}\int_0^1\log^3(y)\tan^{-1}(y)\,dy\overset{IBP}{=}-\frac{3\pi}{4}$$

$$-\frac{1}{2}\left\{\int_0^1\frac{y\log^3(y)}{1+y^2}dy-3\int_0^1\frac{y\log^2(y)}{1+y^2}dy+6\int_0^1\frac{y\log(y)}{1+y^2}dy-6\int_0^1\frac{y}{1+y^2}dy\right\}$$

$$=-\frac{3\pi}{4}-\frac{1}{2}\left\{\frac{1}{16}\int_0^1\frac{\log^3(f)}{1+f}df-\frac{3}{8}\int_0^1\frac{\log^2(f)}{1+f}df+\frac{3}{2}\int_0^1\frac{f\log(f)}{1+f}df-3\log(2)\right\}$$

$$=-\frac{3\pi}{4}$$

$$-\frac{1}{2}\left\{\frac{1}{16}\sum_{k=0}^{\infty}(-1)^k\int_0^1 f^k\log^3(f)\,df-\frac{3}{8}\sum_{k=0}^{\infty}(-1)^k\int_0^1 f^k\log^2(f)\,df\right.$$

$$\left.+\frac{3}{2}\sum_{k=0}^{\infty}(-1)^k\int_0^1 f^k\log(f)\,df-3\log(2)\right\}$$

$$=-\frac{3\pi}{4}-\frac{1}{2}\left\{-\frac{3}{8}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^4}-\frac{3}{4}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^3}-\frac{3}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^2}-3\log(2)\right\}$$

$$=-\frac{3\pi}{4}-\frac{1}{2}\left\{-\frac{3}{8}\eta(4)-\frac{3}{4}\eta(3)-\frac{3}{2}\eta(2)-3\log(2)\right\}$$

$$=-\frac{3\pi}{4}-\frac{1}{2}\left\{-\frac{21}{64}\zeta(4)-\frac{9}{16}\zeta(3)-\frac{3}{4}\zeta(2)-3\log(2)\right\}$$

$$=\frac{9}{32}\zeta(3)+\frac{7\pi^4}{3840}+\frac{\pi^2}{16}-\frac{3\pi}{4}+\frac{3}{2}\log(2)$$

Solution 2 by Togrul Ehmedov-Azerbaijan

$$I=\int_0^1\int_0^1\int_0^1\sum_{x,y,z}\log(x)\tan^{-1}(xyz)\,dx\,dy\,dz=3\int_0^1\int_0^1\int_0^1\log(x)\tan^{-1}(xyz)\,dx\,dy\,dz$$

$$\frac{I}{3}=\int_0^1\int_0^1\int_0^1\log(x)\tan^{-1}(xyz)\,dx\,dy\,dz=\int_0^1\log(x)\int_0^1\left\{\tan^{-1}(xy)-\frac{\log(1+x^2y^2)}{2xy}\right\}dy\,dx$$

$$=\int_0^1\int_0^1\log(x)\tan^{-1}(xy)\,dy\,dx-\frac{1}{2}\int_0^1\int_0^1\frac{\log(x)\log(1+x^2y^2)}{xy}dy\,dx$$

$$=\int_0^1\log(x)\tan^{-1}(x)\,dx-\frac{1}{2}\int_0^1\frac{\log(x)\log(1+x^2)}{x}dx$$

$$-\frac{1}{2}\int_0^1\int_0^1\frac{\log(x)\log(1+x^2y^2)}{xy}dy\,dx=I_1-\frac{1}{2}I_2-\frac{1}{2}I_3$$

$$\int_0^1\log(x)\tan^{-1}(x)\,dx=-\frac{\pi}{4}+\frac{1}{2}\log(2)+\frac{\pi^2}{48}$$

$$\int_0^1\frac{\log(x)\log(1+x^2)}{x}dx=\frac{1}{4}\int_0^1\frac{\log(x)\log(1+x)}{x}dx=-\frac{3}{16}\zeta(3)$$

$$\int_0^1\int_0^1\frac{\log(x)\log(1+x^2y^2)}{xy}dy\,dx=-\frac{1}{2}\int_0^1\frac{\log^2(x)\log(1+x^2)}{x}dx$$

$$=-\frac{1}{16}\int_0^1\frac{\log^2(x)\log(1+x)}{x}dx=-\frac{7\pi^4}{5760}$$

$$\frac{I}{3}=I_1-\frac{1}{2}I_2-\frac{1}{2}I_3=-\frac{\pi}{4}+\frac{1}{2}\log(2)+\frac{\pi^2}{48}+\frac{3}{32}\zeta(3)+\frac{7\pi^4}{11520}$$

$$I=\frac{9}{32}\zeta(3)+\frac{7\pi^4}{3840}+\frac{\pi^2}{16}-\frac{3\pi}{4}+\frac{3}{2}\log(2)$$