ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^1\int_0^1 \frac{e^{x+y}-1}{e^{x+y}+1}dxdy = 4\chi_2(e^2) - 4Li_2(e) + \frac{\pi^2}{6} - 1$$

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$$\sigma = \int_0^1\int_0^1 \frac{e^{x+y}-1}{e^{x+y}+1}dxdy$$

$$subtitation\ \{e^{x+y} = t, dt = tdx, t[e^{1+y}; e^y]\}$$

$$\sigma = \int_0^1\int_{e^y}^{e^y+1} \frac{t-1}{t(t+1)}dtdy = \int_0^1\int_{e^y}^{e^{1+y}}\left(\frac{2}{1+t} - \frac{1}{t}\right)dtdy =$$

$$= \int_0^1 [\,2\ln(1+t) - \ln(t)]_{e^y}^{e^{1+y}}dy =$$

$$= 2\int_0^1 \ln(1+e^{1+y})dy - 2\int_0^1 \ln(1+e^y)dy - \int_0^1 dy =$$

$$= -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1 e^{n(1+y)}dy + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1 e^{ny}dy - 1 =$$

$$= 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\frac{e^{ny}}{n}\right]_0^1 - 2\sum_{n=1}^{\infty}\frac{(-1)^n e^n}{n}\left[\frac{e^{ny}}{n}\right]_0^1 - 1 =$$

$$= 2\sum_{n=1}^{\infty}\frac{(-1)^n e^n}{n^2} - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} - 2\sum_{n=1}^{\infty}\frac{(-1)^n e^{2n}}{n^2} + 2\sum_{n=1}^{\infty}\frac{(-1)^n e^n}{n^2} - 1 =$$

$$= 4Li_2(-e) + 2\eta(2) - 2Li_2(-e^2) - 1$$

Notes:

$$Polylogarithm\ function: Li_a(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^a}, |z| \le 1$$

$$Li_a(z^2) = \frac{1}{2^{1-a}}\left(Li_a(z) + Li_a(-z)\right)$$

$$Legendre\ chi\ function: \chi_a(z) = \sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)^a} = \frac{1}{2}\left(Li_a(z) - Li_a(-z)\right)$$

$$\sigma = 4Li_2(-e) - 4Li_2(e) - 4Li_2(-e) + 4\chi_2(e^2) + \frac{\pi^2}{6} - 1 =$$

$$= 4\chi_2(e^2) - 4Li_2(e) + \frac{\pi^2}{6} - 1$$